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Derivatives and their application

Textbook

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МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
НАЦІОНАЛЬНИЙ ТЕХНІЧНИЙ УНІВЕРСИТЕТ
"ДНІПРОВСЬКА ПОЛІТЕХНІКА"



Похідні та їх застосування

Навчальний посібник

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Навчальний посібник містить основні теоретичні положення стосовно розділу "Похідні та їх застосування" курсу вищої математики. Дано вичерпний аналіз основним способам знаходження похідних та їх використання у подальших задачах. Наведено вичерпні розв'язки достатньої кількості типових задач. Містить понад 20 варіантів індивідуальних завдань з різних тем курсу «Похідна та її застосування». Орієнтований на організацію системної підготовки та самопідготовки. Розраховано на студентів перших та других курсів всіх спеціальностей технічних вищих навчальних закладів денної, вечірньої, заочної та дистанційної форм навчання, іноземних студентів, та студентів спеціальностей з викладанням математики англійською мовою.

The textbook contains the main theoretical provisions of the section "Derivatives and their applications" of the course of higher mathematics. A comprehensive analysis of the main methods of calculation derivatives and their using in further problems are given. Comprehensive solutions to a sufficient number of typical problems are given. The textbook contains more than 20 variants of individual tasks on various topics of the course "Derivatives and their applications". It is focused on the organization of systematic training and self-training. It is designed for the first and second year students of all specialties of technical higher educational institutions of full-time, evening, correspondence and distance learning, foreign students, and students who learn mathematics in English.

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Introduction

The manual “Derivatives and their application” is intended for independent study of the corresponding section of higher mathematics. Each topic of the section contains, in the most accessible form, information from the theory, instructions for solving problems according to a type, and analyzed test cases. Tasks with answers are given for the independent work of students. Particular attention is paid to the study of functions and the application of the derivative to the solution of applied problems.

1. A derivative of a Function

1.1. Definition of a Derivative

Let some function $y = f(x)$ be defined in the interval $(a; b)$. We provide to any x from this interval the derivative increment Δx . The difference $\Delta y = f(x + \Delta x) - f(x)$ is called the increment of the function at the point x .

If the difference quotient $\Delta y/\Delta x$ has a limit as $\Delta x \rightarrow 0$, this limit is called the derivative or differential coefficient of the function $y = f(x)$ with respect to x and we write

$$y' = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If a function has a finite derivative at a point x , it is called differentiable at this point.

The process of the derivative calculation is called differentiation. We denote the derivative $y'; f'(x); \frac{dy}{dx}; \frac{df}{dx}$.

1.2. Basic Rules of Differentiation

Let $u = u(x); v = v(x)$ and $w = w(x)$ be differentiable functions that depend on x and c be a constant. The following relationships hold true:

$$(c)' = 0; \quad (1)$$

$$(u \pm v)' = u' \pm v'; \quad (2)$$

$$(u \cdot v)' = u'v + v'u; \quad (3)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}; \quad (4a)$$

$$\left(\frac{c}{v}\right)' = -\frac{cv'}{v^2}; \quad (4b)$$

$$\left(\frac{u}{c}\right)' = \frac{u'}{c}; \quad (4c)$$

$$(u \cdot v \cdot w)' = u'vw + v'uw + w'uv; \quad (5)$$

$$(cu)' = cu'. \quad (6)$$

1.3. Formulas for Differentiation for the Basic Elementary Functions (Derivative Table)

$y = x^n,$	$y' = nx^{n-1};$	(7)
$y = \sqrt{x},$	$y' = \frac{1}{2\sqrt{x}};$	(8)
$y = \log_a x,$	$y' = \frac{1}{x \ln a};$	(9)
$y = \ln x$	$y' = \frac{1}{x}$	(10)
$y = a^x$	$y' = a^x \ln a$	(11)
$y = e^x$	$y' = e^x$	(12)
$y = \sin x$	$y' = \cos x$	(13)
$y = \cos x$	$y' = -\sin x$	(14)
$y = \operatorname{tg} x$	$y' = \frac{1}{\cos^2 x}$	(15)
$y = \operatorname{ctg} x$	$y' = -\frac{1}{\sin^2 x}$	(16)
$y = \arcsin x$	$y' = \frac{1}{\sqrt{1-x^2}}$	(17)
$y = \arccos x$	$y' = -\frac{1}{\sqrt{1-x^2}}$	(18)
$y = \operatorname{arctg} x$	$y' = \frac{1}{1+x^2}$	(19)
$y = \operatorname{arcctg} x$	$y' = -\frac{1}{1+x^2}$	(20)

Let's consider examples of finding derivatives using formulas (1-20).

Example 1. $y = 10x^5 + 5x^4 - 3$.

Solution. Using (2), (1), (6), (7) we obtain

$$y' = (10x^5 + 5x^4 - 3)' = (10x^5)' + (5x^4)' - 3' = 10(x^5)' + 5(x^4)' - 3' = 10 \cdot 5x^4 + 5 \cdot 4x^3 - 0 = 50x^4 + 20x^3.$$

Example 2. $y = \frac{4}{x^3} - \frac{3}{\sqrt[3]{x^2}}.$

Solution. As $\frac{1}{a^n} = a^{-n}$ and $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ we obtain

$$y' = \left(\frac{4}{x^3} - \frac{3}{\sqrt[3]{x^2}} \right)' = (4x^{-3} - 3x^{-\frac{2}{3}})' = 4 \cdot (-3)x^{-3-1} - 3 \cdot \left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}.$$

These derivatives could be calculated using (4a). We offer to do it by yourself.

Example 3. $y = \ln x \cdot \arctg x$.

Solution. Using (3), (10), (19) we obtain

$$y' = (\ln x \cdot \arctg x)' = (\ln x)' \cdot \arctg x + \ln x \cdot (\arctg x)' = \frac{1}{x} \arctg x + \ln x \frac{1}{1+x^2}.$$

Example 4. $y = \frac{\sin x}{3^x}$.

Solution Using (4), (13), (11) we obtain

$$y' = \left(\frac{\sin x}{3^x} \right)' = \frac{(\sin x)' \cdot 3^x - (3^x)' \cdot \sin x}{(3^x)^2} = \frac{\cos x \cdot 3^x - 3^x \ln 3 \cdot \sin x}{3^{2x}}.$$

Individual Task 1

Find the derivatives of the functions using (7-8) from the Derivative Table.

Variant 1

1. $y = 3x^4 + x^9 - 3x + 5$	2. $y = \frac{7}{x} + \frac{2}{x^5} - \frac{3}{x^7}$	3. $y = 3\sqrt[3]{x} + 2\sqrt[9]{x} - 3\sqrt{x}$
4. $y = \frac{7}{\sqrt{x}} + \frac{2}{\sqrt[5]{x}} - \frac{3}{\sqrt[7]{x}}$	5. $y = 7x^4 + \sqrt[5]{x} - \frac{3}{x} + \frac{6}{\sqrt[3]{x}}$	6. $y = 12 - x - 2\sqrt{x} - \frac{3}{x^3} + \frac{9}{\sqrt{x}}$

Variant 2

1. $y = 4x^3 + 7x^7 - 3x^5 + 1$	2. $y = \frac{7}{2x^2} + \frac{2}{x^4} - \frac{13}{x^8}$	3. $y = 8\sqrt[5]{x} + 2\sqrt[6]{x} - 7\sqrt{x}$
4. $y = \frac{1}{5\sqrt{x}} + \frac{2}{\sqrt[7]{x}} - \frac{3}{2\sqrt[5]{x}}$	5. $y = x^5 + 2\sqrt[8]{x} - \frac{1}{5x} + \frac{6}{\sqrt{x}}$	6. $y = 2x^9 - \sqrt[3]{x} - \frac{1}{3x^3} + \frac{9}{\sqrt[4]{x}}$

Variant 3

1. $y = 6x^{13} + 3x^2 - 2x^3 - 4$	2. $y = \frac{1}{3x^3} + \frac{2}{x^6} - \frac{3}{x^{78}}$	3. $y = \sqrt[4]{x} + 6\sqrt[6]{x} - \sqrt{x} + \sqrt[12]{x}$
4. $y = \frac{2}{7\sqrt{x}} + \frac{1}{7\sqrt[17]{x}} - \frac{3}{\sqrt[9]{x}}$	5. $y = x^{15} + 3\sqrt[3]{x} - \frac{15}{x^5} + \frac{6}{\sqrt[7]{x}}$	6. $y = 5x^4 - \sqrt[3]{x} - \frac{31}{x^{13}} + \frac{9}{\sqrt[8]{x}}$

Variant 4

1. $y = 9x^{10} + 3x^{12} - 7x^3 - 4$	2. $y = \frac{1}{3x^{32}} + \frac{2}{x^{16}} - \frac{3}{x^7}$	3. $y = \sqrt[24]{x} + \sqrt[5]{x} - \sqrt{x} + \sqrt[9]{x}$
4. $y = \frac{4}{7\sqrt[92]{x}} - \frac{3}{\sqrt[6]{x}} + \frac{14}{\sqrt[3]{x}}$	5. $y = 2x - 3\sqrt[5]{x} - \frac{5}{x^5} + \frac{6}{\sqrt[5]{x}}$	6. $y = x^{12} - \sqrt[4]{x} - \frac{3}{x^3} + \frac{9}{\sqrt[3]{x}}$

Variant 5

1. $y = 6x^6 + 3x^{21} - 7x^9 - 4$	2. $y = \frac{3}{x^{22}} + \frac{8}{x^{26}} - \frac{3}{x^8}$	3. $y = \sqrt[14]{x} + \sqrt[6]{x} - \sqrt{x} + \sqrt[19]{x}$
4. $y = \frac{7}{\sqrt{x}} + \frac{4}{\sqrt[12]{x}} - \frac{3}{\sqrt[5]{x}}$	5. $y = x^{15} + \sqrt[4]{x} - \frac{4}{x^4} + \frac{6}{\sqrt[4]{x}}$	6. $y = 7x^2 - \sqrt[14]{x} - \frac{13}{x^8} + \frac{9}{\sqrt[9]{x}}$

Variant 6

1. $y = \frac{4}{5}x^{11} + 3x^2 - 7x^{32} - 9$	2. $y = \frac{5}{3x^{42}} + \frac{2}{x^6} - \frac{3}{x^{17}}$	3. $y = \sqrt[8]{x} + \sqrt[12]{x} - 7 + \sqrt[7]{x}$
4. $y = \frac{4}{\sqrt[2]{x}} - \frac{3}{\sqrt[62]{x}} + \frac{4}{\sqrt[13]{x}}$	5. $y = -x + \sqrt[4]{x} - \frac{4}{x^4} + \frac{8}{\sqrt[4]{x}}$	6. $y = 5x^2 - \sqrt[7]{x} - \frac{3}{x^7} + \frac{9}{\sqrt[7]{x}}$

Variant 7

1. $y = \frac{4}{5}x^5 + 7x^{12} - 2x^2 - 8$	2. $y = \frac{5}{x^{52}} + \frac{2}{5x^{16}} - \frac{3}{2x}$	3. $y = \sqrt[3]{x} + 3\sqrt{x} + 7\sqrt[13]{x}$
4. $y = \frac{4}{\sqrt{x}} + \frac{9}{\sqrt[9]{x}} - \frac{6}{\sqrt[12]{x}}$	5. $y = x^7 - \sqrt[7]{x} - \frac{4}{x^7} + \frac{7}{\sqrt[7]{x}}$	6. $y = x - 2\sqrt[6]{x} - \frac{3}{x^6} + \frac{6}{\sqrt[6]{x}}$

Variant 8

1. $y = \frac{1}{4}x^4 + 2x^4 - 7x + 1$	2. $y = \frac{4}{x^2} + \frac{1}{3x^3} - \frac{2}{6x^6}$	3. $y = 6\sqrt[6]{x} + 2\sqrt[9]{x} - \frac{1}{2}\sqrt{x}$
4. $y = \frac{1}{2\sqrt{x}} + \frac{6}{\sqrt[6]{x}} - \frac{4}{\sqrt[7]{x}}$	5. $y = x^6 + \sqrt[6]{x} - \frac{5}{x^6} - \frac{1}{\sqrt[6]{x}}$	6. $y = 4x - \sqrt[8]{x} - \frac{3}{x^{13}} + \frac{9}{\sqrt[7]{x}}$

Variant 9

1. $y = 1 - 4x^{13} + 3x^7 - 7x$	2. $y = \frac{7}{x^{12}} + \frac{1}{2x} - \frac{3}{x^7}$	3. $y = 8\sqrt[8]{x} + 2\sqrt[7]{x} - \sqrt{2x}$
4. $y = 2 - \frac{2}{\sqrt{x}} + \frac{7}{\sqrt[7]{x}} - \frac{5}{2\sqrt[5]{x}}$	5. $y = 7x^3 + \sqrt[8]{x} - \frac{1}{x} + \frac{6}{\sqrt[3]{x}}$	6. $y = 2x^9 - \sqrt[3]{x} - \frac{3}{x^3} + \frac{4}{\sqrt[4]{x}}$

Variant 10

1. $y = 6 - 2x^{13} + 3x^5 - 2x$	2. $y = \frac{9}{x^9} + \frac{2}{x} - \frac{3}{x^7}$	3. $y = 5\sqrt[5]{x} + 6\sqrt[6]{x} - 7\sqrt{x}$
4. $y = \frac{2}{\sqrt{x}} + \frac{8}{\sqrt[84]{x}} - \frac{13}{\sqrt[9]{x}}$	5. $y = x^5 + \sqrt[5]{x} - \frac{15}{x^5} + \frac{6}{\sqrt[15]{x}}$	6. $y = 5x^{14} - \sqrt[3]{x} - \frac{3}{x^2} + \frac{1}{\sqrt[18]{x}}$

Variant 11

1. $y = 6 - 4x^{16} + 3x - 7x^2$	2. $y = \frac{1}{2x^2} + \frac{8}{x^6} - \frac{13}{x^{81}}$	3. $y = \sqrt[4]{x} + \frac{1}{5}\sqrt[5]{x} - 2\sqrt[9]{x}$
4. $y = \frac{4}{\sqrt[8]{x}} - \frac{3}{\sqrt[3]{x}} + \frac{16}{\sqrt[4]{x}}$	5. $y = 7\sqrt[14]{x} - \frac{8}{x^7} + \frac{6}{\sqrt[3]{x}}$	6. $y = \frac{x^5}{5} - 3\sqrt[4]{x} - \frac{13}{x^{18}} - \frac{3}{\sqrt[9]{x}}$

Variant 12

1. $y = 3 - x^6 + 2x^9 - 3x$	2. $y = 7 - \frac{2}{x} + \frac{2}{x^{15}} - \frac{7}{x^7}$	3. $y = 4\sqrt[14]{x} + 7\sqrt[7]{x} - 2\sqrt{x}$
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4. $y = \frac{2}{\sqrt{x}} + \frac{6}{\sqrt[12]{x}} - \frac{3}{\sqrt[3]{x}}$	5. $y = 5x + \sqrt[15]{x} - \frac{3}{x} + \frac{6}{\sqrt[9]{x}}$	6. $y = x - 2\sqrt[5]{x} - \frac{3}{x^2} + \frac{9}{\sqrt[8]{x}}$
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Variant 13

1. $y = 4 + 2x^{10} + x^7 - 3x^2$	2. $y = \frac{7}{x^{12}} + \frac{1}{2x^4} - \frac{1}{5x^5}$	3. $y = 8 - \sqrt[5]{x} + 2\sqrt[16]{x} - 4\sqrt{x}$
4. $y = \frac{12}{5\sqrt{x}} + \frac{2}{\sqrt[75]{x}} - \frac{5}{\sqrt[25]{x}}$	5. $y = 7x^{15} + \sqrt[4]{x} - \frac{1}{x} + \frac{6}{\sqrt[7]{x}}$	6. $y = x^3 - \sqrt[3]{x} - \frac{1}{3x^3} + \frac{3}{\sqrt[3]{x}}$

Variant 14

1. $y = 2x^3 + 3x^{12} - 2x - 4$	2. $y = \frac{1}{3x^{13}} + \frac{2}{x^5} - \frac{3}{x^8}$	3. $y = \sqrt[14]{x} + \sqrt[16]{x} - \sqrt{x} + \sqrt[7]{x}$
4. $y = \frac{5}{\sqrt{x}} + \frac{7}{\sqrt[14]{x}} - \frac{3}{\sqrt[9]{x}}$	5. $y = 2x + \sqrt[9]{x} - \frac{11}{x^{55}} + \frac{6}{\sqrt[6]{x}}$	6. $y = 2\sqrt[5]{x} - \frac{33}{x^{33}} + \frac{9}{\sqrt[8]{x}}$

Variant 15

1. $y = 2x^2 + 3x^{13} - 7x^7 - 4$	2. $y = \frac{1}{2x^2} + \frac{24}{x^8} - \frac{7}{x^7}$	3. $y = \frac{\sqrt[5]{x}}{2} - \frac{\sqrt{x}}{2} + \sqrt[17]{x}$
4. $y = \frac{4}{\sqrt{x}} - \frac{3}{\sqrt[3]{x}} + \frac{14}{\sqrt[78]{x}}$	5. $y = x - \sqrt[5]{x} - \frac{5}{x^5} + \frac{6}{\sqrt[5]{x}}$	6. $y = \frac{x^2}{2} - \sqrt[42]{x} - \frac{3}{x^{13}} + \frac{9}{\sqrt[9]{x}}$

Variant 16

1. $y = \frac{x^6}{6} + 3x^7 - 2x^{19} - 4$	2. $y = \frac{2}{x^2} + \frac{8}{x^{24}} - \frac{3}{x^7}$	3. $y = -6\sqrt[6]{x} - 3\sqrt{x} + \sqrt[25]{x}$
4. $y = -\frac{6}{\sqrt[6]{x}} - \frac{3}{\sqrt[3]{x}} + \frac{14}{\sqrt[7]{x}}$	5. $y = x^8 + \frac{\sqrt[4]{x}}{4} - \frac{12}{x^4} + \frac{2}{\sqrt[6]{x}}$	6. $y = x^3 - \frac{\sqrt[14]{x}}{2} - \frac{3}{x^5} + \frac{3}{\sqrt[9]{x}}$

Variant 17

1. $y = 5x^{10} + 4x^2 - 7x^{13} - 9$	2. $y = \frac{3}{x^3} + \frac{1}{2x^{14}} - \frac{3}{x^7}$	3. $y = \sqrt[4]{x} + \frac{2}{3}\sqrt[52]{x} - 9\sqrt{x}$
4. $y = -\frac{4}{\sqrt[22]{x}} - \frac{13}{\sqrt[39]{x}} + \frac{4}{\sqrt[12]{x}}$	5. $y = -2\sqrt[24]{x} - \frac{4}{x^{14}} + \frac{8}{\sqrt[34]{x}}$	6. $y = -2\sqrt[3]{x} - \frac{3}{x^{72}} + \frac{9}{\sqrt[17]{x}}$

Variant 18

1. $y = 9 - 2x^{15} + 7x^{12} - 2x^{12}$	2. $y = \frac{1}{x^5} + \frac{2}{5x^6} - \frac{3}{2x^{17}}$	3. $y = \frac{\sqrt[7]{x}}{7} + 4\sqrt[4]{x} - 7\sqrt[9]{x}$
4. $y = \frac{6}{\sqrt[9]{x}} - \frac{9}{\sqrt[9]{x}} + \frac{5}{\sqrt[5]{x}}$	5. $y = 7 + \sqrt[5]{x} - \frac{4}{x^9} + \frac{5}{\sqrt[8]{x}}$	6. $y = x - 6\sqrt[6]{x} - \frac{5}{x^3} + \frac{4}{\sqrt[4]{x}}$

Variant 19

1. $y = \frac{1}{3}x^3 + 5x^5 - 9 - x$	2. $y = \frac{9}{x^{11}} + \frac{1}{3x^{53}} - \frac{7}{6x^6}$	3. $y = \sqrt[6]{x} + 2\sqrt[12]{x} - \frac{1}{9}\sqrt{x}$
4. $y = \frac{5}{\sqrt{x}} + \frac{6}{\sqrt[12]{x}} - \frac{45}{\sqrt[45]{x}}$	5. $y = x^6 + \sqrt[6]{x} - \frac{5}{x^{12}} - \frac{1}{\sqrt[6]{x}}$	6. $y = x - \sqrt[81]{x} - \frac{3}{x^5} + \frac{5}{\sqrt[17]{x}}$

Variant 20

1. $y = 1 - 2x^3 + 3x^{17} - 12x$	2. $y = \frac{4}{x^2} + \frac{1}{5x} - \frac{3}{x^{27}}$	3. $y = 8 - \sqrt[18]{x} + 2\sqrt[7]{x} - \sqrt{5x}$
4. $y = \frac{22}{\sqrt{x}} + \frac{7}{\sqrt[5]{x}} - \frac{25}{2\sqrt[75]{x}}$	5. $y = 4 - \sqrt[6]{x} - \frac{4}{x} + \frac{6}{\sqrt[9]{x}}$	6. $y = x^{19} - \sqrt[3]{x} - \frac{3}{x^{31}} + \frac{4}{\sqrt[45]{x}}$

Individual Task 2

Obtain the derivatives of the functions using (9–20) from the Derivative Table.

Variant 1

1. $y = 7 \ln x - 2 \sin x$	2. $y = 8 \cos x - 5e^x$	3. $y = 7 \cos x + 9 \arcsin x$
4. $y = 7 \operatorname{ctgx} + 5^x + \arcsin x$	5. $y = 3 \arctgx - 4 \operatorname{tgx}$	6. $y = 2 \arccos x - 3 \operatorname{arccctgx}$

Variant 2

1. $y = 5 \arctgx - 7 \operatorname{ctgx}$	2. $y = 7 \operatorname{tgx} + 2^x + 9 \arcsin x$	3. $y = 2 \sin x + 3 \arccos x$
4. $y = 4 \cos x - 6e^x$	5. $y = 8 \ln x - 5 \sin x$	6. $y = 2 \arcsin x - \arctgx$

Variant 3

1. $y = 15 \arctgx - 6e^x$	2. $y = 2 \arcsin x + 7^x$	3. $y = 8 \sin x - 6 \arctgx$
4. $y = 8 \ln x - 12 \operatorname{ctgx}$	5. $y = 14 \cos x - 3 \sin x$	6. $y = 7 \operatorname{tgx} + 7 \arccos x$

Variant 4

1. $y = 32 \arcsin x - 6 \operatorname{ctgx}$	2. $y = 15 \arctgx - 7 \sin x$	3. $y = 4 \cos x + 15 \arccos x$
4. $y = 22 \operatorname{tgx} - 6e^x$	5. $y = 8 \sin x + 9^x$	6. $y = 4 \ln x + 2 \arctgx$

Variant 5

1. $y = 4 \cos x + 3 \arccos x$	2. $y = 9 \arcsin x - 4 \operatorname{ctgx}$	3. $y = 6 \arctgx + 8 \sin x$
4. $y = 3 \sin x + 22e^x$	5. $y = 18 \ln x + 15^x$	6. $y = 72 \operatorname{tgx} - 23 \arctgx$

Variant 6

1. $y = 7 \operatorname{tgx} - 5 \cdot 2^x$	2. $y = 19 \arcsin x - 2 \operatorname{tgx}$	3. $y = 18 \operatorname{ctgx} - 2 \operatorname{arccctgx}$
4. $y = 4 \sin x - 7 \ln x$	5. $y = 6 \arctgx + 9e^x$	6. $y = 4 \cos x - 7 \arcsin x$

Variant 7

1. $y = 2 \operatorname{ctgx} - 5 \cdot \arccos x$	2. $y = 19 \sin x - 12 \operatorname{tgx}$	3. $y = 8 \operatorname{tgx} - 2 \arctgx$
4. $y = 8 \cos x - 7e^x$	5. $y = 5 \arcsin x + 4e^x$	6. $y = 55 \ln x - 7 \operatorname{arccctgx}$

Variant 8

1. $y = 5 \arctgx + 3e^x$	2. $y = 5 \ln x - 7 \arcsin x$	3. $y = 18 \sin x - 4 \cdot 9^x$
4. $y = 3 \operatorname{ctgx} - 2 \arctgx$	5. $y = 9 \cos x - 52 \operatorname{ctgx}$	6. $y = 2 \operatorname{tgx} - 7 \cdot \arccos x$

Variant 9

1. $y = 2 \cos x - 8 \operatorname{ctgx}$	2. $y = 11 \sin x - 4 \cdot 8^x$	3. $y = 5 \ln x - 7 \arcsin x$
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4. $y = 12tgx - 6arccos x$	5. $y = 25arctgx + 6e^x$	6. $y = 7ctgx - 21arctgx$
Variant 10		
1. $y = 6tgx - 2arctgx$	2. $y = 99arcctgx + 12e^x$	3. $y = 12ctgx - 16arccos x$
4. $y = 6\ln x - 4arcsin x$	5. $y = 11 - \sin x - 45^x$	6. $y = 2arcsin x - 9ctgx$
Variant 11		
1. $y = 25\ln x + 5\sin x$	2. $y = 18\cos x - 5e^x$	3. $y = 3\cos x + 15arcsin x$
4. $y = ctgx + 12^x + arcsin x$	5. $y = 7arctgx - 14tgx$	6. $y = 2arccos x - 3arcctgx$
Variant 12		
1. $y = 15arctgx - 2ctgx$	2. $y = 7tgx + 7^x + 9arcsin x$	3. $y = arcsin x + 3arccos x$
4. $y = 4\cos x - 6e^x$	5. $y = 8\ln x - 5\sin x$	6. $y = 2\sin x - arctgx$
Variant 13		
1. $y = 8arctgx - 9e^x$	2. $y = 2arcsin x + 2 \cdot 4^x$	3. $y = 18\sin x - 6 + arctgx$
4. $y = 27\ln x - 12 - ctgx$	5. $y = 4\cos x - 3\sin x$	6. $y = 3tgx + 7arccos x$
Variant 14		
1. $y = \pi arcsin x - 6 + ctgx$	2. $y = 15arctgx - 7\sin x$	3. $y = 3\cos x + 15arccos x$
4. $y = 11tgx - 6e^x$	5. $y = 8\sin x + 6 \cdot 9^x$	6. $y = 4\ln x + 2arcctgx$
Variant 15		
1. $y = \pi \cos x + 3arccos x$	2. $y = 9arcsin x - 4ctgx$	3. $y = 6arcctgx + 8\sin x$
4. $y = 3 + \sin x + 22e^x$	5. $y = 9\ln x + 2 \cdot 5^x$	6. $y = 24tgx - 2 - 3arctgx$
Variant 16		
1. $y = 7ctgx - 4 \cdot 7^x$	2. $y = \pi arcsin x - 4ctgx$	3. $y = 1 + 8tgx - 2arctgx$
4. $y = 2\sin x - 12\ln x$	5. $y = 7arcctgx + 3e^x$	6. $y = 14\cos x - 4arcsin x$
Variant 17		
1. $y = 12tgx - \pi arccos x$	2. $y = 9\sin x - 12ctgx$	3. $y = 2ctgx - 29arctgx$
4. $y = 16\cos x - 4e^x$	5. $y = 5arcsin x + 45e^x$	6. $y = 6\ln x - 4arcctgx$
Variant 18		
1. $y = 56arcctgx + 38e^x$	2. $y = 52\ln x - 11arcsin x$	3. $y = 9\sin x - 7 \cdot 6^x$
4. $y = 3 + 2tgx - 21arctgx$	5. $y = 7\cos x - 9tgx$	6. $y = 2ctgx - 7 + 2arccos x$
Variant 19		
1. $y = \pi \cos x - 8ctgx$	2. $y = 12\sin x - 4 \cdot 3^x$	3. $y = 15\ln x - 2arcsin x$
4. $y = 13tgx - 7arccos x$	5. $y = 15arcctgx + 88e^x$	6. $y = 17ctgx - 21arctgx$
Variant 20		
1. $y = 16tgx - 4 - 2arcctgx$	2. $y = 9arctgx + \pi e^x$	3. $y = 6ctgx - 17arccos x$
4. $y = 16\ln x - 4arcsin x$	5. $y = 11\sin x - 2 \cdot 45^x$	6. $y = 2arccos x - 9ctgx$

Individual Task 3

Obtain the derivatives of the functions using basic rules of differentiation (1-6).

Variant 1

1. $y = 16tgx \cdot arcctgx$	2. $y = 9arctgx \cdot e^x$	3. $y = 6ctgx \cdot arc \cos x$
4. $y = \frac{16 \ln x}{\arcsin x}$	5. $y = \frac{11 \sin x}{45^x}$	6. $y = \frac{2 \arccos x}{-9ctgx}$

Variant 2

1. $y = \frac{16tgx}{arcctgx}$	2. $y = \frac{9arctgx}{e^x}$	3. $y = \frac{6ctgx}{arc \cos x}$
4. $y = 16 \ln x \cdot \arcsin x$	5. $y = 11 \sin x \cdot 45^x$	6. $y = 2 \arccos x \cdot ctgx$

Variant 3

1. $y = \pi \cos x \cdot ctgx$	2. $y = 12 \sin x \cdot 3^x$	3. $y = 15 \ln x \cdot \arcsin x$
4. $y = \frac{13tgx}{\arccos x}$	5. $y = \frac{15arcctgx}{e^x}$	6. $y = \frac{17ctgx}{arctgx}$

Variant 4

1. $y = \frac{3 \cos x}{ctgx}$	2. $y = \frac{12 \sin x}{3^x}$	3. $y = \frac{15 \ln x}{2 \arcsin x}$
4. $y = 13tgx \cdot \arccos x$	5. $y = 15arcctgx \cdot e^x$	6. $y = 17ctgx \cdot arctgx$

Variant 5

1. $y = 56arcctgx \cdot e^x$	2. $y = 52 \ln x \cdot \arcsin x$	3. $y = 9 \sin x \cdot 6^x$
4. $y = \frac{2tgx}{arctgx}$	5. $y = \frac{7 \cos x}{9tgx}$	6. $y = \frac{2ctgx}{7 \arccos x}$

Variant 6

1. $y = \frac{56arcctgx}{38e^x}$	2. $y = \frac{2 \ln x}{\arcsin x}$	3. $y = \frac{9 \sin x}{7 \cdot 6^x}$
4. $y = 2tgx \cdot arctgx$	5. $y = 7 \cos x \cdot tgx$	6. $y = 2ctgx \cdot \arccos x$

Variant 7

1. $y = 12tgx \cdot \arccos x$	2. $y = 9 \sin x \cdot ctgx$	3. $y = 2ctgx \cdot arctgx$
4. $y = \frac{16e^x}{\cos x}$	5. $y = \frac{5 \arcsin x}{e^x}$	6. $y = \frac{6 \ln x}{arcctgx}$

Variant 8

1. $y = \frac{12tgx}{\pi \arccos x}$	2. $y = \frac{9 \sin x}{\ln x}$	3. $y = \frac{2ctgx}{9arctgx}$
4. $y = 16 \cos x \cdot e^x$	5. $y = 5 \arcsin x \cdot e^x$	6. $y = 6 \ln x \cdot arcctgx$

Variant 9

1. $y = 7ctgx \cdot 7^x$	2. $y = \pi \arcsin x \cdot ctgx$	3. $y = 8tgx \cdot arctgx$
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4. $y = \frac{2 \sin x}{\ln x}$	5. $y = \frac{7 \operatorname{arctg} x}{e^x}$	6. $y = \frac{14 \cos x}{\arcsin x}$
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Variant 10

1. $y = \frac{7 \operatorname{ctg} x}{7^x}$	2. $y = \frac{\pi \arcsin x}{\operatorname{ctg} x}$	3. $y = \frac{2 \operatorname{arctg} x}{\operatorname{tg} x}$
4. $y = 2 \sin x \cdot \ln x$	5. $y = 7 \operatorname{arctg} x \cdot e^x$	6. $y = 14 \cos x \cdot \arcsin x$

Variant 11

1. $y = \pi \cos x \cdot \arccos x$	2. $y = 9 \arcsin x \cdot \operatorname{ctg} x$	3. $y = 6 \operatorname{arctg} x \cdot \sin x$
4. $y = \frac{3 + \sin x}{e^x}$	5. $y = \frac{9 \ln x}{\operatorname{arctg} x}$	6. $y = \frac{24 \cdot 3^x}{\operatorname{tg} x}$

Variant 12

1. $y = \frac{\pi \cos x}{3 \arccos x}$	2. $y = \frac{9 \arcsin x}{\operatorname{ctg} x} - 4 \operatorname{ctg} x$	3. $y = \frac{6 \operatorname{arctg} x}{\sin x}$
4. $y = (3 + \sin x) \cdot e^x$	5. $y = 9 \ln x \cdot 5^x$	6. $y = 24 \operatorname{tg} x \cdot \operatorname{arctg} x$

Variant 13

1. $y = 6 \operatorname{ctg} x \cdot \operatorname{arctg} x$	2. $y = 4 \arccos x \cdot 8^x$	3. $y = 2 \operatorname{tg} x \cdot \arcsin x$
4. $y = \frac{6 \arccos x}{\ln x}$	5. $y = \frac{7 + \sin x}{5^x}$	6. $y = \frac{2 \operatorname{arctg} x}{9 - \operatorname{ctg} x}$

Variant 14

1. $y = \frac{16 + \operatorname{tg} x}{\operatorname{arctg} x}$	2. $y = \frac{9 + \operatorname{arctg} x}{2 - e^x}$	3. $y = \frac{6 \cos x}{\arcsin x}$
4. $y = (16 + \ln x) \cdot \arcsin x$	5. $y = 12 \sin x \cdot 4^x$	6. $y = (2 + \arccos x) \cdot \operatorname{tg} x$

Variant 15

1. $y = \frac{3 - \cos x}{\operatorname{tg} x}$	2. $y = \frac{2 \sin x}{6^x}$	3. $y = \frac{5 \ln x}{2 + \arcsin x}$
4. $y = (3 - \operatorname{tg} x) \cdot \arccos x$	5. $y = 7 \operatorname{arctg} x \cdot e^x$	6. $y = (7 - \operatorname{ctg} x) \cdot \operatorname{arctg} x$

Variant 16

1. $y = (6 + \operatorname{arctg} x) \cdot e^x$	2. $y = 2 \ln x \cdot (5 + \arcsin x)$	3. $y = 4 \sin x \cdot 9^x$
4. $y = \frac{2 - \operatorname{tg} x}{\operatorname{arctg} x}$	5. $y = \frac{7 \cos x}{9 - \operatorname{tg} x}$	6. $y = \frac{2 \operatorname{ctg} x}{7 \ln x}$

Variant 17

1. $y = \frac{5 - 6 \operatorname{arctg} x}{3e^x}$	2. $y = \frac{2 + \ln x}{\arccos x}$	3. $y = \frac{\ln x}{2 \cdot 6^x}$
4. $y = (2 + \sin x) \cdot \operatorname{arctg} x$	5. $y = (7 - \cos x) \cdot \operatorname{tg} x$	6. $y = 4 \operatorname{ctg} x \cdot \sqrt[4]{x}$

Variant 18

1. $y = 12 \operatorname{tg} x \cdot \ln x$	2. $y = 9 \sin x \cdot 8^x$	3. $y = (2 + \operatorname{ctg} x) \cdot \operatorname{arctg} x$
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4. $y = \frac{6 + e^x}{\cos x}$	5. $y = \frac{5 - \arcsin x}{4e^x}$	6. $y = \frac{6 \ln x}{5 + \operatorname{arctg} x}$
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Variant 19

1. $y = \frac{7 \operatorname{ctg} x}{2 - \arccos x}$	2. $y = \frac{8 \sin x}{\ln x + 5}$	3. $y = \frac{2 - \operatorname{ctg} x}{9 \operatorname{arctg} x}$
4. $y = 16 \cos x \cdot (2 - e^x)$	5. $y = 5 \arcsin x \cdot \ln x$	6. $y = 6\sqrt[5]{x} \cdot \operatorname{arc} \operatorname{ctg} x$

Variant 20

1. $y = (7 + \operatorname{ctg} x) \cdot 7^x$	2. $y = (\pi - \arcsin x) \cdot \operatorname{ctg} x$	3. $y = (8 + \operatorname{tg} x) \cdot \operatorname{arc} \operatorname{ctg} x$
4. $y = \frac{2 - \sin x}{5 \ln x}$	5. $y = \frac{7 \operatorname{arctg} x}{e^x - 4}$	6. $y = \frac{4 \cos x}{4 - \arcsin x}$

1.4. Derivative of a Function of a Function (composite function)

If $y = f(u)$ has a derivative at the point u and $u = g(x)$ has a derivative at the point x then a function of a function $y = f(g(x))$ has a derivative at the point x , and

$$y' = f'(u)g'(x) \text{ or } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (21)$$

In other words, its derivative is obtained by differentiating the outer function f with respect to g and the inner function with respect to x and multiplying the two derivatives

We obtain the derivatives of the functions using the rule of differentiation of a composite function.

Example 1. $y = (3x^5 + 4)^8$.

Solution. The outer function is a power function then we use (7) and (2)

$$y' = 8(3x^5 + 4)^{8-1} \cdot (3x^5 + 4)' = 8(3x^5 + 4)^7 \cdot 15x^4 = 120x^4(3x^5 + 4)^7.$$

Example 2. $y = \cos^4 x$.

Solution. The outer function is a power function and the inner function is a trigonometric function. We use (7) и (14) and obtain

$$y' = (\cos^4 x)' = 4 \cos^3 x \cdot (\cos x)' = 4 \cos^3 x (-\sin x) = -4 \cos^3 x \sin x.$$

Example 3. $y = \sqrt{x^5 + \arcsin x + 3}$.

Solution. Using (8), (2), (7), (17), (1) we obtain

$$y' = (\sqrt{x^5 + \arcsin x + 3})' = \frac{1}{2\sqrt{x^5 + \arcsin x + 3}} \cdot (x^5 + \arcsin x + 3)' =$$

$$\frac{1}{2\sqrt{x^5 + \arcsin x + 3}} \cdot (5x^4 + \frac{1}{\sqrt{1-x^2}}).$$

Example 4. $y = 5^{\operatorname{tg} x} \cdot e^{\sin x}$.

Solution. Using (3), (11), (15), (12), (13) we obtain

$$y = (5^{tgx} e^{\sin x})' = (5^{tgx})' e^{\sin x} + 5^{tgx} (e^{\sin x})' = 5^{tgx} \ln 5 \cdot (tgx)' e^{\sin x} + 5^{tgx} e^{\sin x} (\sin x)' =$$

$$= 5^{tgx} \ln 5 \cdot \frac{1}{\cos^2 x} e^{\sin x} + 5^{tgx} e^{\sin x} \cos x = 5^{tgx} e^{\sin x} (\ln 5 \cdot \frac{1}{\cos^2 x} + \cos x).$$

Example 5. $y = \frac{\sqrt{ctgx}}{\log_3^4 x}.$

Solution. Using (4), (8), (16), (7), (9) we obtain

$$y' = \left(\frac{\sqrt{ctgx}}{\log_3^4 x} \right)' = \frac{(\sqrt{ctgx})' \log_3^4 x - \sqrt{ctgx} (\log_3^4 x)'}{(\log_3^4 x)^2} =$$

$$= \frac{\frac{1}{2\sqrt{ctgx}} \left(-\frac{1}{\sin^2 x} \right) \log_3^4 x - \sqrt{ctgx} (4 \log_3^3 x \cdot \frac{1}{x \ln 3})}{(\log_3^4 x)^2} =$$

$$= \frac{-\log_3^3 x \left(\frac{\log_3 x}{2\sqrt{ctgx} \sin^2 x} + \frac{4\sqrt{ctgx}}{x \ln 3} \right)}{\log_3^8 x} = -\frac{\frac{\log_3 x}{2\sqrt{ctgx} \sin^2 x} + \frac{4\sqrt{ctgx}}{x \ln 3}}{\log_3^5 x}.$$

Example 6. $y = \ln^4(\operatorname{arccctg} \sqrt{5x^2 + e^{3x}} - 1).$

Solution. Using (7), (10), (20), (8), (2), (7), (12), (1) we obtain

$$y' = (\ln^4(\operatorname{arccctg} \sqrt{5x^2 + e^{3x}} - 1))' = 4 \ln^3(\operatorname{arccctg} \sqrt{5x^2 + e^{3x}} - 1) \cdot$$

$$\cdot \frac{1}{\operatorname{arccctg} \sqrt{5x^2 + e^{3x}} - 1} \left(-\frac{1}{1 + (\sqrt{5x^2 + e^{3x}} - 1)^2} \right) \frac{1}{2\sqrt{5x^2 + e^{3x}} - 1} (10x + e^{3x} 3).$$

Individual Task 4

Obtain the derivatives of the composite functions using (21)

Variant 1

1. $y = 16tg3x + \operatorname{arccctg}^2 x$	2. $y = 9 \operatorname{arctg} x^3 + e^{3x}$	3. $y = ctg \sqrt{x} + \arccos(7x^3)$
4. $y = 6\sqrt{\ln x} \cdot \arcsin \sqrt{x}$	5. $y = 11 \sin^2 x \cdot 5^{2x}$	6. $y = 2 \arccos x^2 \cdot ctg^2 x$
7. $y = \frac{16 \ln^3 x^2}{\sqrt{\arcsin x}}$	8. $y = \frac{11 \sqrt{\sin x}}{4 + 5^x}$	9. $y = \frac{2 \arccos x^3}{9 ctg^3 x}$

Variant 2

1. $y = \cos 5x + 7 \operatorname{arctg}^3 x$	2. $y = 9 \sin x^3 + 5^{3x}$	3. $y = 7tg \sqrt[4]{x} + \ln(4x^3)$
4. $y = 6\sqrt{ctg 2x} \cdot \arcsin x^6$	5. $y = 11 \arcsin^2 x \cdot e^{2x}$	6. $y = 2 \cos x^2 \cdot tg^3 x$
7. $y = \frac{6 \ln^2 3x^2}{\sqrt{\arcsin 2x}}$	8. $y = \frac{\sqrt{\sin 5x}}{5 \ln^2 x}$	9. $y = \frac{2 \operatorname{arccctg} 4x^2}{9tg^4 x}$

Variant 3

1. $y = 2e^{5x} + \operatorname{ctg}^3 x$	2. $y = 9\sin x^3 + \cos^3 x$	3. $y = 2\operatorname{ctg}\sqrt{x} + \operatorname{tg}(4x^3)$
4. $y = 3\sqrt{\ln 2x} \cdot \arcsin \sqrt{x}$	5. $y = 11\arcsin^2 x \cdot \cos x^2$	6. $y = 2\operatorname{ctg}x^2 \cdot \arcsin^3 x$
7. $y = \frac{\sqrt{\ln 3x^2}}{3\arccos\sqrt[4]{2x}}$	8. $y = \frac{\cos\sqrt{5x}}{5\ln x^5}$	9. $y = \frac{3\operatorname{arctg}3x^3}{\operatorname{tg}^3 x}$

Variant 4

1. $y = 2\cos 3x + 9\ln^2 x$	2. $y = 9\arcsin x^3 + \operatorname{tg} 3x$	3. $y = 6\operatorname{ctg}\sqrt{x} + \arccos(7x^3)$
4. $y = 6\sqrt{\ln x} \cdot \arcsin \sqrt{x}$	5. $y = 11\sin^2 x \cdot 5^{2x}$	6. $y = 2\arccos x^2 \cdot \operatorname{ctg}^2 x$
7. $y = \frac{16\ln^3 x^2}{\sqrt{\arcsin x}}$	8. $y = \frac{11\sqrt{\sin x}}{4 + 5^x}$	9. $y = \frac{2\arccos x^3}{9\operatorname{ctg}^3 x}$

Variant 5

1. $y = 12\cos 5x + 3\operatorname{arctg}^3 x$	2. $y = 9\sin x^3 + 5^{3x}$	3. $y = 7e^{\sqrt[4]{x}} + \operatorname{arctg}(7x^3)$
4. $y = 4\sqrt{\sin 4x} \cdot \arccos x^4$	5. $y = 7\arcsin^7 x \cdot e^{7x}$	6. $y = 4\cos x^4 \cdot \ln^4 x$
7. $y = \frac{\operatorname{ctg}^2 2x^5}{\sqrt{x}}$	8. $y = \frac{\sqrt{\cos 3x}}{3\ln^4 x}$	9. $y = \frac{\operatorname{arcctg} 7x^7}{7\operatorname{tg}^7 x}$

Variant 6

1. $y = 5\ln 5x + \operatorname{ctg}^5 x$	2. $y = 4\sin x^4 + \cos^4 x$	3. $y = 6\operatorname{ctg}\sqrt[6]{x} + \operatorname{tg}(6x^6)$
4. $y = 3\sqrt[3]{\ln 3x} \cdot \arcsin \sqrt[3]{x}$	5. $y = 8\arcsin^8 x \cdot \cos x^8$	6. $y = 3\operatorname{ctg}x^5 \cdot \arcsin^3 x$
7. $y = \frac{\sqrt[5]{\sin 3x^5}}{6\ln \sqrt[4]{2x}}$	8. $y = \frac{\operatorname{ctg}\sqrt{5x}}{5e^{x^5}}$	9. $y = \frac{3\operatorname{arctg} 5x^7}{\operatorname{ctg}^4 x}$

Variant 7

1. $y = 16\ln 3x + \cos^2 x$	2. $y = 2\operatorname{arctg}x^4 + 4e^{3x}$	3. $y = 7\operatorname{tg}\sqrt{x} + \arccos(2\sqrt{x^3})$
4. $y = 6\sqrt{\sin x} \cdot \arcsin \sqrt{x}$	5. $y = 8\sin^4 x \cdot 4^{4x}$	6. $y = 6\arccos x^6 \cdot \operatorname{ctg}^6 x$
7. $y = \frac{9\ln^9 x}{\sqrt{\arcsin x}}$	8. $y = \frac{8\sqrt{\sin x}}{5 + e^{2x}}$	9. $y = \frac{20\cos x^3}{\operatorname{ctg}^3 x}$

Variant 8

1. $y = 2\operatorname{ctg} 8x + \arcsin^8 x$	2. $y = 4\sin x^4 + \ln 4x$	3. $y = 4\operatorname{tg}\sqrt[7]{x} + \ln(7x^7)$
4. $y = 2\sqrt{\operatorname{tg} 6x} \cdot \arcsin(2x^2)$	5. $y = \arccos^7 x \cdot e^{4x}$	6. $y = 5\cos x^5 \cdot \operatorname{tg}^7 x$
7. $y = \frac{\ln^3(3x^2)}{\sqrt{\arcsin 5x}}$	8. $y = \frac{\sqrt{\operatorname{tg} 6x}}{6\cos^2 x}$	9. $y = \frac{2\operatorname{arctg}(4x^4)}{\operatorname{tg}^4 x}$

Variant 9

1. $y = 5e^{7x} + \operatorname{tg}^6 x$	2. $y = 3\sin x^5 + \cos^5 x$	3. $y = 42\operatorname{tg}\sqrt[3]{x} + \cos(3x^4)$
4. $y = 3\sqrt{\sin 5x} \cdot e^{4\sqrt{x}}$	5. $y = 2\operatorname{arctg}^8 x \cdot \cos x^5$	6. $y = 2\cos(3x^2) \cdot \ln^3 x$

7. $y = \frac{\sqrt{\ln 3x^2}}{3\arccos\sqrt[4]{2x}}$	8. $y = \frac{\cos\sqrt{5x}}{5\ln x^5}$	9. $y = \frac{3\arctg 3x^3}{tg^3 x}$
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Variant 10

1. $y = 4\ln 4x + \arcsin^4 x$	2. $y = 2\arctg x^3 + tg^3 x$	3. $y = 5ctg\sqrt[5]{x} + \arccos(2x^5)$
4. $y = 4\sqrt{\sin x} \cdot \arcsin \sqrt{x}$	5. $y = 7\sin^7(7x) \cdot 7^{2x}$	6. $y = 3\cos x^2 \cdot \arctg^3 2x$
7. $y = \frac{6\sqrt{\ln^3 x^2}}{\cos 5x^4}$	8. $y = \frac{\sqrt{\sin x}}{e^{3x^2}}$	9. $y = \frac{\cos x^8}{8\arctg^3 x}$

Variant 11

1. $y = 4\cos 4x + 2\arctg^3 x$	2. $y = \sin 9x^3 + 2^{3x}$	3. $y = 7\sin\sqrt[4]{x} + \sqrt[3]{\arctg x}$
4. $y = 5\sqrt{\sin 3x} \cdot \arccos(5x^4)$	5. $y = 4\arcsin^8 x \cdot e^{2x}$	6. $y = 4\arccos x^7 \cdot \ln^3 x$
7. $y = \frac{tg^2 x}{\sqrt{2x^5}}$	8. $y = \frac{\sqrt{\cos 4x^2}}{4\ln^4 x}$	9. $y = \frac{\arctg 3x^3}{3ctg^7 x}$

Variant 12

1. $y = 5e^{5x} + tg^5 x$	2. $y = 2\sin x^8 + 4\cos^2 x$	3. $y = 3ctg\sqrt[3]{x} + 3tg(3x^3)$
4. $y = 4\sqrt[4]{\ln 4x} \cdot \arcsin\sqrt[4]{x}$	5. $y = 3\arcsin^6 x \cdot \cos x^2$	6. $y = 7\cos x^5 \cdot \ln^3 x$
7. $y = \frac{\sqrt[5]{\arcsin x^5}}{\sqrt[4]{2x}}$	8. $y = \frac{tg\sqrt{7x}}{2^{x^5}}$	9. $y = \frac{3\arcsin 5x^7}{\arctg^4 x}$

Variant 13

1. $y = e^{3x} + 7\cos^2 x$	2. $y = 4\arctg x^4 + 14\ln 4x$	3. $y = ctg\sqrt{x} + \cos(2\sqrt{x^5})$
4. $y = 7\sqrt{\sin x} \cdot \arctg\sqrt{x}$	5. $y = 5\sin^4 x \cdot \cos(4x^4)$	6. $y = 5\arccos x^5 \cdot tg^5 x$
7. $y = \frac{4\ln^3 x}{\sqrt{\arcsin 3x}}$	8. $y = \frac{8\sqrt{\sin 2x}}{\cos x^2}$	9. $y = 3\frac{\cos x^3}{tg^3 x}$

Variant 14

1. $y = 8ctg 2x + \arcsin^5 x$	2. $y = 3\sin x^3 + 4\ln 4x$	3. $y = 4tg\sqrt[3]{x} + 5\ln(7x^4)$
4. $y = \sqrt{tg 3x} \cdot \arcsin(2x^6)$	5. $y = \arccos^6 x \cdot e^{5x}$	6. $y = 5\cos(3x^3) \cdot tg^7 x$
7. $y = \frac{\ln^4(3x^7)}{\sqrt{\arcsin 9x}}$	8. $y = \frac{\sqrt{ctg 5x}}{5\cos^3 x}$	9. $y = \frac{2\arctg(3x^4)}{tg^2 x}$

Variant 15

1. $y = 6e^{3x} + 8tg^2 x$	2. $y = 8\sin x^5 + 5\cos^7 x$	3. $y = 3tg\sqrt[4]{x} + \cos(4x^3)$
4. $y = 2\sqrt{\sin 7x} \cdot e^{2\sqrt{x}}$	5. $y = 9\arctg^4 x \cdot \cos(4x^5)$	6. $y = 8\cos(5x^4) \cdot \ln^3 x$
7. $y = \frac{\sqrt{\ln 3x^6}}{2\arccos\sqrt[5]{x}}$	8. $y = \frac{\arccos\sqrt{x}}{5tgx^5}$	9. $y = \frac{3\arctg(4x^7)}{ctg^3 x}$

Variant 16

1. $y = 4^{4x} + 6\arcsin^6 x$	2. $y = 2\arctg x^6 + 5tg^3 x$	3. $y = 2ctg\sqrt[3]{x} + 3\arccos(5x^2)$
4. $y = 4\sqrt{\ln x} \cdot \arcsin\sqrt{x}$	5. $y = 7\arcsin^4 x \cdot 4^{7x}$	6. $y = 9\cos x^{12} \cdot \arctg^3 3x$
7. $y = \frac{\sqrt{ctg^3 x^2}}{2\cos 9x^4}$	8. $y = \frac{\sqrt{\sin 3x}}{6e^{x^2}}$	9. $y = \frac{\cos x^5}{6\arctg^2 x}$

Variant 17

1. $y = \cos 5x + 12\arctg^5 x$	2. $y = 3\sin 2x^3 + 9^{3x}$	3. $y = 2\sin \sqrt[8]{x} + 5\sqrt[3]{\arccotg x}$
4. $y = 15\sqrt{\sin 3x} \cdot \cos(5x^4)$	5. $y = 3\arcsin^6 x \cdot e^{7x}$	6. $y = 4\arcsin x^5 \cdot \ln^3 x$
7. $y = \frac{tg^4 x}{3\sqrt{x^5}}$	8. $y = \frac{\sqrt{\cos x^2}}{5\ln^4 x}$	9. $y = \frac{\arctg 2x^3}{3tg^3 x}$

Variant 18

1. $y = 2\ln 5x + 4tg^4 x$	2. $y = 6\sin(5x^5) + 4\cos^3 x$	3. $y = 2tg\sqrt[3]{x} + 5ctg(7x^4)$
4. $y = 2\sqrt[3]{\ln 7x} \cdot \arcsin \sqrt[4]{x}$	5. $y = 3\arcsin^5 x \cdot \cos(6x^2)$	6. $y = 7\arccos x^5 \cdot \ln^3 x$
7. $y = \frac{\sqrt[5]{\sin(7x^5)}}{6\sqrt[4]{x}}$	8. $y = \frac{tg\sqrt{9x+6}}{e^{x^5}}$	9. $y = \frac{\arcsin 5x^7}{ctg^4 x}$

Variant 19

1. $y = 4\cos 4x + \arctg^5 x$	2. $y = 2\arctg x^3 + 7ctg^3 x$	3. $y = 7tg\sqrt[5]{x} + \arccos(7x^8)$
4. $y = 4\sqrt{\sin 4x} \cdot \arcsin\sqrt{x}$	5. $y = 2\sin^7(4x) \cdot 7^{2x}$	6. $y = 3\cos x^5 \cdot \arccotg^2 4x$
7. $y = \frac{3\sqrt{\ln^2 x^3}}{2\cos 4x^2}$	8. $y = \frac{\sqrt{\sin 5x}}{6e^{3x^2}}$	9. $y = \frac{\cos x^3}{6\arccotg^5 x}$

Variant 20

1. $y = 3\cos 4x^5 + 2\arctg^3 x$	2. $y = 7\sin 2x^3 + 6e^{3x}$	3. $y = 7\cos \sqrt[4]{x} + 4\sqrt[3]{\arctg x}$
4. $y = 8\sqrt{\sin 3x} \cdot \cos(5x^4)$	5. $y = 4\sin^8 x \cdot ctg 2x^6$	6. $y = 4\cos x^7 \cdot \ln^3 x$
7. $y = \frac{ctg^2 x}{\sqrt{2x^3}}$	8. $y = \frac{\sqrt{\cos 7x^2}}{9\ln^4 x}$	9. $y = \frac{\arctg 5x^3}{3ctg^7 x}$

1.5. Derivative of the Inverse Function

If the function $y = f(x)$ is differentiable in a given interval (a, b) where $f'(x) \neq 0$, then the inverse function $x = g(y)$ possesses a derivative at all points in the corresponding interval and

$$g'(y) = \frac{1}{f'(x)}, \quad (y'_x = \frac{1}{x'_y}). \quad (22)$$

Example 1. Obtain the derivative y'_x of $x = y^5 - 3y^2$.

Solution. $x'_y = 5y^4 - 6y$. Using (22), $y'_x = \frac{1}{5y^4 - 6y}$.

Example 2. Obtain the derivative y'_x of $x = y \ln y + \sin y$.

Solution. $x'_y = y'_y \ln y + y(\ln y)'_y + \cos y = \ln y + 1 + \cos y$. $y'_x = \frac{1}{\ln y + 1 + \cos y}$.

1.6. Derivative of the Implicit Function

Let $y(x)$ be an implicit function of x , $F(x, y) = 0$. There are three points to obtain the derivative y' .

1. Differentiate all terms of the equation with respect to x .
2. Carry out the differentiation.
3. Solve for y' .

Example 1. Obtain the derivative y'_x of $x^2 + y^5 = 25$.

Solution. Differentiate all terms of the equation $x^2 + y^5 = 25$ with respect to x and obtain $2x + 5y^4 y' = 0 \Rightarrow y' = -\frac{2x}{5y^4}$.

Example 2. Obtain the derivative y'_x of $x^3 + 10x^2 y^3 + y^3 = 10x^2$.

Solution. Differentiate all terms of the equation with respect to x
 $3x^2 + 10(2xy^3 + x^2 3y^2 y') + 3y^2 y' = 10 \cdot 2x$; $3x^2 + 20xy^3 + 30x^2 y^2 y' + 3y^2 y' = 20x$;
 $3y^2 y'(10x^2 + 1) = 20x - 3x^2 - 20xy^3$; $y' = \frac{20x - 3x^2 - 20xy^3}{3y^2(10x^2 + 1)} = \frac{x(20 - 3x - 20y^3)}{3y^2(10x^2 + 1)}$.

Example 3. Obtain the derivative y'_x of $\sin(xy) + \cos(\frac{x}{y}) = \tan(x + y)$.

Solution. Differentiate all terms of the equation with respect to x

$$\cos(xy) \cdot (xy)' - \sin(\frac{x}{y}) \cdot (\frac{x}{y})' = \frac{1}{\cos^2(x + y)} (x + y)';$$

$$\cos(xy) \cdot (y + xy') - \sin(\frac{x}{y}) \cdot \frac{y - xy'}{y^2} = \frac{1}{\cos^2(x + y)} (1 + y');$$

$$y \cos(xy) + xy' \cos(xy) - \frac{1}{y} \sin(\frac{x}{y}) + \frac{xy'}{y^2} \sin(\frac{x}{y}) = \frac{1}{\cos^2(x + y)} + \frac{y'}{\cos^2(x + y)};$$

$$y' \left(x \cos(xy) + \frac{x}{y^2} \sin(\frac{x}{y}) - \frac{1}{\cos^2(x + y)} \right) = \frac{1}{\cos^2(x + y)} - y \cos(xy) + \frac{1}{y} \sin(\frac{x}{y});$$

$$y' = \frac{\frac{1}{\cos^2(x+y)} - y \cos(xy) + \frac{1}{y} \sin\left(\frac{x}{y}\right)}{\left(x \cos(xy) + \frac{x}{y^2} \sin\left(\frac{x}{y}\right) - \frac{1}{\cos^2(x+y)}\right)}.$$

1.7. Derivative of the Parametric Function

Let $y(x)$ be a parametric function that is given by parametric equations $x = \phi(t)$, $y = \varphi(t)$ where $\phi(t)$, $\varphi(t)$ are differentiable functions of t , $\phi(t) \neq 0$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\varphi'(t)}{\phi'(t)} \quad \text{or} \quad y'_x = \frac{y'_t}{x'_t}. \quad (23)$$

Example 1. Obtain y'_x of $\begin{cases} y = 3\sin^3 t; \\ x = 2\cos^3 t. \end{cases}$

Solution. As $y'_t = (3\sin^3 t)'_t = 3 \cdot 3\sin^2 t \cos t = 9\sin^2 t \cos t$;
 $x'_t = (2\cos^3 t)'_t = 2 \cdot 3\cos^2 t(-\sin t) = -6\sin t \cos^2 t$, then using (27),

$$y'_x = \frac{9\sin^2 t \cos t}{-6\sin t \cos^2 t} = -\frac{3 \sin t}{2 \cos t} = -\frac{3}{2} \tan t.$$

Example 2. Obtain y'_x of $\begin{cases} y = t^2, \\ x = t^3 + 1. \end{cases}$

Solution. $y'_t = 2t$; $x'_t = 3t^2$. Using (23), $y'_x = \frac{2}{3} \frac{t}{t^2} = \frac{2}{3t}$.

Example 3. Obtain y'_x of $\begin{cases} y = e^{2t} \sin^2 t, \\ x = e^{2t} \cos^2 t. \end{cases}$

Solution.

$$\begin{aligned} y'_t &= (e^{2t} \sin^2 t)'_t = \{\text{according to (3)}\} = (e^{2t})' \sin^2 t + e^{2t} (\sin^2 t)' = \\ &= e^{2t} 2\sin^2 t + e^{2t} 2\sin t \cos t; \\ x'_t &= (e^{2t} \cos^2 t)'_t = (e^{2t})' \cos^2 t + e^{2t} (\cos^2 t)' = e^{2t} 2\cos^2 t - e^{2t} 2\sin t \cos t. \end{aligned}$$

$$\text{Using (22)} \quad y'_x = \frac{2e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t}{2e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t} = \frac{2\sin^2 t + \sin 2t}{2\cos^2 t - \sin 2t}.$$

1.8. Logarithmic Differentiation. Derivative of the Exponential and Power Function

Certain functions may be more easily differentiated by expressing them logarithmically first. There are some cases to use logarithmic differentiation.

1. We need to differentiate the product of three or more functions or a fraction whose numerator and denominator contain the product of functions.

2. We need to differentiate the exponential and power function $y = V(x)^{U(x)}$.

The function $y = V(x)^{U(x)}$ can be differentiated using the formula

$$y' = (V(x)^{U(x)})' = V^U \ln V \cdot U' + UV^{U-1}V'. \quad (24)$$

They are often mistaken, considering the function $y = V(x)^{U(x)}$ only exponential or only exponential.

Example 1. Obtain the derivative of the function $y = \frac{(x+1)^2 \cdot \cos x^2 \cdot 3^{2x}}{\sqrt{5x-1} \cdot \operatorname{tg} x}$.

Solution. Let us logarithm this expression using the rules

$$\log_c |a \cdot b| = \log_c |a| + \log_c |b|; \quad (25)$$

$$\log_c \left| \frac{a}{b} \right| = \log_c |a| - \log_c |b|; \quad (26)$$

$$\log_c |a^n| = n \log_c |a|. \quad (27)$$

We obtain $\ln y = \ln \frac{(x+1)^2 \cdot \cos x^2 \cdot 3^{2x}}{\sqrt{5x-1} \cdot \operatorname{tg} x} = \ln((x+1)^2 \cdot \cos x^2 \cdot 3^{2x}) - \ln(\sqrt{5x-1} \cdot \operatorname{tg} x);$

$$\ln y = \ln(x+1)^2 + \ln \cos x^2 + \ln 3^{2x} - (\ln \sqrt{5x-1} + \ln \operatorname{tg} x);$$

$$\ln y = 2 \ln(x+1) + \ln \cos x^2 + 2x \ln 3 - \frac{1}{2} \ln(5x-1) - \ln \operatorname{tg} x.$$

Differentiate both parts of the expression obtained

$$\frac{1}{y} y' = \frac{2}{x+1} + \frac{1}{\cos x^2} (-\sin x^2) 2x + 2 \ln 3 - \frac{1}{2} \frac{5}{5x-1} - \frac{1}{\operatorname{tg} x} \frac{1}{\cos^2 x};$$

$$y' = y \left(\frac{2}{x+1} + \frac{1}{\cos x^2} (-\sin x^2) 2x + 2 \ln 3 - \frac{1}{2} \frac{5}{5x-1} - \frac{1}{\operatorname{tg} x} \frac{1}{\cos^2 x} \right);$$

$$y' = \frac{(x+1)^2 \cos x^2 \cdot 3^{2x}}{\sqrt{5x-1} \cdot \operatorname{tg} x} \left(\frac{2}{x+1} - \frac{\sin x^2}{\cos x^2} 2x + 2 \ln 3 - \frac{5}{2(5x-1)} - \frac{1}{\operatorname{tg} x \cos^2 x} \right).$$

Example 2. Obtain the derivative of the function $y = (\operatorname{tg} x)^{\sin x}$.

Solution. Use logarithmic differentiation and obtain

$$\ln y = \ln (\operatorname{tg} x)^{\sin x} = (\text{from 24}) = \sin x \ln(\operatorname{tg} x);$$

$$\frac{1}{y} y' = (\sin x \ln(\operatorname{tg} x))' = (\sin x)' \ln(\operatorname{tg} x) + \sin x (\ln(\operatorname{tg} x))' =$$

$$= \cos x \ln(\operatorname{tg} x) + \sin x \frac{1}{\operatorname{tg} x} \frac{1}{\cos^2 x} = \cos x \ln(\operatorname{tg} x) + \sin x \frac{\cos x}{\sin x} \frac{1}{\cos^2 x};$$

$$y' = y(\cos x \ln(\operatorname{tg} x) + \frac{1}{\cos x}) = (\operatorname{tg} x)^{\sin x} (\cos x \ln(\operatorname{tg} x) + \frac{1}{\cos x}).$$

Example 3. Obtain the derivative of the function $y = (\arcsin 5x)^{\sqrt{2x^2+3x}}$.

Solution. Use (25). Then $V(x) = \arcsin 5x$; $U(x) = \sqrt{2x^2+3x}$.

$$\begin{aligned} y' &= (\arcsin 5x)^{\sqrt{2x^2+3x}} \cdot \ln(\arcsin 5x) \left(\sqrt{2x^2+3x} \right)' + \\ &+ \sqrt{2x^2+3x} (\arcsin 5x)^{\sqrt{2x^2+3x}-1} \cdot (\arcsin 5x)' = \\ &= (\arcsin 5x)^{\sqrt{2x^2+3x}} \cdot \ln(\arcsin 5x) \cdot \frac{4x+3}{2\sqrt{2x^2+3x}} + \\ &+ \sqrt{2x^2+3x} \cdot (\arcsin 5x)^{\sqrt{2x^2+3x}-1} \cdot \frac{5}{\sqrt{1-(5x)^2}}. \end{aligned}$$

Individual Task 5

Obtain the derivatives of the inverse, implicit, parametric, exponential and power functions using (22-24)

Variant 1

1. $x = \sin^3 y + \cos^3 y$	2. $\begin{cases} x = t^2 - 2t, \\ y = t^2 + 2t; \end{cases}$	3. $y = (\cos^3 4x)^{\sqrt[3]{2x}}$
4. $x \sin y - \cos y + \cos 2y = 0$	5. $\begin{cases} x = 2(t - \sin t), \\ y = 2(1 - \cos t); \end{cases}$	6. $y = (\operatorname{tg} x)^{\sqrt{\cos 2x}}$

Variant 2

1. $x = e^{\sin y}$	2. $\begin{cases} x = \sin t^2, \\ y = \cos^2 t; \end{cases}$	3. $y = (\operatorname{tg} 3x)^{\cos x}$
4. $xe^y - \cos x + \cos 2y = 0$	5. $\begin{cases} x = 2 \sin^3 t, \\ y = 2 \cos^3 t; \end{cases}$	6. $y = (\ln x)^{\sqrt{\sin 5x}}$

Variant 3

1. $x = \sqrt{y} + \cos y^2$	2. $\begin{cases} x = \sqrt{t^2 - 2t}, \\ y = \ln t; \end{cases}$	3. $y = (\operatorname{ctg}^2 x)^{\sqrt{x}}$
4. $y \sin 2x - \sqrt{x} \cos y + \cos(xy) = 0$	5. $\begin{cases} x = t \sin t, \\ y = t - \cos t; \end{cases}$	6. $y = (\operatorname{arctg} x)^{\ln x}$

Variant 4

1. $x = \operatorname{ctg} y + 5 \cos y$	2. $\begin{cases} x = 3 \sin^2 t, \\ y = \cos t^3; \end{cases}$	3. $y = (\sin 4x)^{5x+2}$
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4. $2e^y - y \cos x + x \cos 2y = 0$	5. $\begin{cases} x = \operatorname{ctg} 7t^2, \\ y = 2 \cos^3 t; \end{cases}$	6. $y = \left(\sqrt[5]{\ln x} \right)^{\operatorname{ctg} x}$
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Variant 5

1. $x = 2 \sin^4 y + 4 \cos^2 y$	2. $\begin{cases} x = \arcsin t^2, \\ y = \ln(t^2 + 2t); \end{cases}$	3. $y = (\cos x)^{2\sqrt[5]{2x}}$
4. $xe^y - \cos x + y \operatorname{tg} 8x = 0$	5. $\begin{cases} x = \sqrt[3]{(t - \sin t)}, \\ y = (1 - 7 \cos t); \end{cases}$	6. $y = (\sin x)^{\sqrt{\arccos 2x}}$

Variant 6

1. $x = y + 5e^{\cos y}$	2. $\begin{cases} x = \arcsin t^2, \\ y = \arccos^2 t; \end{cases}$	3. $y = 5(\operatorname{tg} 3x)^{\arccos x}$
4. $2ye^{xy} - 2 \cos x + \cos 2y = 0$	5. $\begin{cases} x = \arcsin^3 t, \\ y = 2\sqrt{\cos t}; \end{cases}$	6. $y = (\arcsin x)^{\sqrt{\sin 5x}}$

Variant 7

1. $x = 3\sqrt{y} + 5 \operatorname{ctg} y^2$	2. $\begin{cases} x = \sqrt{t^2 - 2t}, \\ y = 3e^t; \end{cases}$	3. $y = (\arctg^2 x)^{\ln x}$
4. $4yx - \sqrt{x} \cos 2y + \cos(xy) = 0$	5. $\begin{cases} x = \sqrt{t \sin t}, \\ y = t - \cos 3t; \end{cases}$	6. $y = (\operatorname{ctg} 3x)^{4 \ln x}$

Variant 8

1. $x = \operatorname{ctg} y + 5 \cos y$	2. $\begin{cases} x = 3 \sin^2 t, \\ y = \cos t^3; \end{cases}$	3. $y = (\sin 4x)^{5x+2}$
4. $2xe^y - \cos(xy) + x \cos 6y = 0$	5. $\begin{cases} x = \operatorname{ctg}^7 t^2, \\ y = 2 \cos 3t; \end{cases}$	6. $y = (\cos x^2)^{3 \operatorname{ctg} x}$

Variant 9

1. $x = 5 \sin^3 y + \cos^3 7y$	2. $\begin{cases} x = (t^2 - 2t)^3, \\ y = \operatorname{tg} 2t; \end{cases}$	3. $y = (\operatorname{ctg} \sqrt{x})^{8 \sin x}$
4. $2x \sin 3y - x \cos y + \operatorname{tg}(2xy) = 0$	5. $\begin{cases} x = \operatorname{ctg} \sqrt{t} \\ y = \sqrt[5]{(1 - \cos t)}; \end{cases}$	6. $y = (\sqrt[4]{\operatorname{tg} 2x})^{\cos x}$

Variant 10

1. $x = e^{\arccos 2y}$	2. $\begin{cases} x = \sin \sqrt{t}, \\ y = \sqrt{\cos t}; \end{cases}$	3. $y = (\operatorname{ctg} 3x)^{4 \cos 4x}$
4. $2ye^x - 3y \cos x + \cos 2y = 0$	5. $\begin{cases} x = \arcsin t, \\ y = \arccos t^2; \end{cases}$	6. $y = (\arccos x)^{x^3}$

Variant 11

1. $x = 3\sqrt{y} + 2\cos^2 y$	2. $\begin{cases} x = \sqrt{3t^3 - 2t}, \\ y = e^t; \end{cases}$	3. $y = (\operatorname{ctg} 3x^2)^{\frac{1}{x}}$
4. $3\sin(xy) - \sqrt{x} \cos y + y \operatorname{tg} x = 0$	5. $\begin{cases} x = \sqrt{t \sin t}, \\ y = (t - \cos t)^2; \end{cases}$	6. $y = (\operatorname{tg} 5x)^{\frac{2}{x}}$

Variant 12

1. $x = 3\operatorname{ctgy} + 5\cos 2y$	2. $\begin{cases} x = 3\sin^4 t, \\ y = 3\cos t^2; \end{cases}$	3. $y = (\cos 3x)^{\operatorname{tg}(5x+2)}$
4. $xe^y - y^2 \cos x + x \ln 5y = 0$	5. $\begin{cases} x = 2\operatorname{tg} 2t^2, \\ y = 2\cos^2 3t; \end{cases}$	6. $y = (x)^{\operatorname{ctgx}^2}$

Variant 13

1. $x = 2e^{2y} + 4\sin^4 y + 4y$	2. $\begin{cases} x = 4\arcsin 2t, \\ y = \cos t^2; \end{cases}$	3. $y = (\sqrt{x})^{\frac{1}{x}}$
4. $9e^{2y} - 2y^2 \operatorname{tg} x + 8y \ln x = 0$	5. $\begin{cases} x = 5\cos \sqrt{x}, \\ y = \cos 6t; \end{cases}$	6. $y = (\ln x)^{\sqrt{\arccos 5x}}$

Variant 14

1. $x = 2\sin y + 5^{\cos y}$	2. $\begin{cases} x = \operatorname{arctg} 2t^2, \\ y = 2\arccos^2 t; \end{cases}$	3. $y = (\operatorname{ctg} 3x)^{\arccos x}$
4. $6^{2xy} - 2y \cos x + \ln 2y = 0$	5. $\begin{cases} x = 3\operatorname{arctg}^3 t, \\ y = 2\sqrt{\sin t}; \end{cases}$	6. $y = (\arccos 2x)^{\sqrt{\sin 4x}}$

Variant 15

1. $x = 3\ln y + 5\operatorname{tg} 3y^2$	2. $\begin{cases} x = 3\cos t, \\ y = 3e^{3t}; \end{cases}$	3. $y = (\operatorname{ctg}^2 x)^{3\ln x}$
4. $5\sqrt{x} \cos 2y + 3\cos(xy) = 0$	5. $\begin{cases} x = t\sqrt{\sin t}, \\ y = t \cos 3t; \end{cases}$	6. $y = (\operatorname{tg} 3x)^{4\sqrt{x}}$

Variant 16

1. $x = 5\operatorname{tgy} + 5\cos 5y$	2. $\begin{cases} x = 4\sin^2 t, \\ y = \sqrt{\cos t^3}; \end{cases}$	3. $y = (x)^{5x+2}$
4. $2x + e^y - \operatorname{tg}(xy) + x \cos 6y = 0$	5. $\begin{cases} x = 3\operatorname{tg}^3 t^2, \\ y = 2\sqrt{t}; \end{cases}$	6. $y = (\ln \sqrt[3]{x})^{2\operatorname{tg} x}$

Variant 17

1. $x = 2y \ln^4 y + 4ctg^2 y$	2. $\begin{cases} x = \ln t, \\ y = ctg(4t^2 + 2t); \end{cases}$	3. $y = (\cos 3x)^{4 \ln x}$
4. $\frac{x}{y} - y \cos x + yctgx = 0$	5. $\begin{cases} x = 4tgt, \\ y = t \cos t; \end{cases}$	6. $y = (\sin x)^{\sqrt{\ln x}}$

Variant 18

1. $x = 2y^4 + 5 \sin 3y$	2. $\begin{cases} x = \sqrt{\sin 2t}, \\ y = 2 \arccos^3 t; \end{cases}$	3. $y = (tg \sqrt[3]{x})^{\cos 3x}$
4. $3x - 3yctgx + \cos(2xy) = 0$	5. $\begin{cases} x = \sqrt[3]{ctgt}, \\ y = 2 \sin \sqrt{t}; \end{cases}$	6. $y = (\arcsin 3x)^{\sqrt{\sin x}}$

Variant 19

1. $x = 3 \frac{1}{\sqrt{y}} + 2ctgy^2$	2. $\begin{cases} x = \ln 2t, \\ y = 2e^t; \end{cases}$	3. $y = (arctg^2 x)^{\sqrt{x}}$
4. $\frac{4y}{x} - \sqrt{x}ctgy + \cos(xy) = 0$	5. $\begin{cases} x = \sqrt{t} \cos t, \\ y = \cos 3t; \end{cases}$	6. $y = \left(\frac{1}{x+2} \right)^{4 \ln x}$

Variant 20

1. $x = 6tgy + 5 \cos y^2$	2. $\begin{cases} x = ctgt, \\ y = \sqrt{\cos t^3}; \end{cases}$	3. $y = (\sin 4x)^{\frac{1}{x}}$
4. $2xe^y - tg(xy) + x \cos y = 0$	5. $\begin{cases} x = \sqrt{ctgt^2}, \\ y = 2 \cos 3t; \end{cases}$	6. $y = \left(\cos \frac{1}{x} \right)^{3ctgx}$

1.9. Geometric, Physical and Mechanical Meaning of the Derivative

1. The derivative of the function $y = f(x)$ for every x equals the slope of the tangent to the graph of this function at a given point, i.e. $f'(x_0) = tg \alpha$, where α is the angle between the tangent line to the graph of the function at the point x_0 and positive direction of the Ox axis (Fig. 1).

Therefore, the equation of the tangent to the graph of the function $y = f(x)$ at the point $M(x_0; y_0)$ is

$$y - y_0 = f'(x_0)(x - x_0). \quad (28)$$

The equation of the normal line to the graph of the function $y = f(x)$ at the same point $M(x_0; y_0)$ could be obtained by the formula:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0). \quad (29)$$

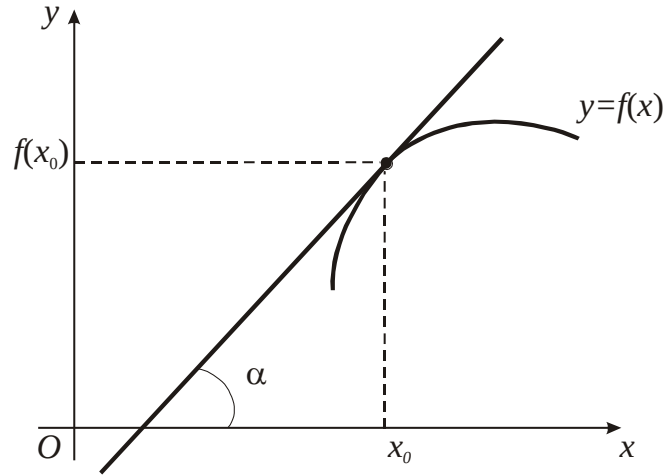


Fig. 1

2. Let the function $y = f(x)$ describe some physical process. The derivative y' is the velocity of this process change. The magnitude of the average velocity is given by $\frac{\Delta y}{\Delta x}$. The instantaneous velocity can be obtained at limiting process as $\Delta x \rightarrow 0$

Hence the instantaneous velocity is $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ and we realize that this limit is the first derivative $f'(x)$.

This is the physical meaning of the derivative.

3. If $S = S(t)$ is the law of motion of a material point then the derivative $S'(t)$ is the velocity v at any instant of time t and the second derivative $S''(t)$ is the instantaneous acceleration a at any instant of time t , i.e.

$$v = S'(t); \quad a = S''(t) = v'(t). \quad (30)$$

This is the mechanical meaning of the derivative.

Example 1. Find the equations of the tangent and the normal lines to the curve $y = \frac{x-1}{x^2+1}$ which pass through the point $x_0 = 0$.

Solution. The ordinate of the tangency point is $y_0 = y(0) = \frac{0-1}{0+1} = -1$, so this point is $M(0; -1)$. Now let us find the derivative:

$$\begin{aligned} y' &= \left(\frac{x-1}{x^2+1} \right)' = \frac{(x-1)'(x^2+1) - (x^2+1)'(x-1)}{(x^2+1)^2} = \frac{(x^2+1) - 2x(x-1)}{(x^2+1)^2} = \\ &= \frac{x^2+1-2x^2+2x}{(x^2+1)^2} = \frac{-x^2+2x+1}{(x^2+1)^2}. \end{aligned}$$

The value of the derivative at point M is $y'(0) = \frac{0+2 \cdot 0+1}{(0+1)^2} = 1$. Using (28) and

(29) for $x_0, y_0, y'(0)$ we obtain:

$y - (-1) = 1(x - 0) \Rightarrow y + 1 = x \Rightarrow y = x - 1$ is the equation of the tangent line;

$y - (-1) = -\frac{1}{1}(x - 0) \Rightarrow y + 1 = -x \Rightarrow y = -x - 1$ is the equation of the normal line.

Example 2. Find the equations of the tangent and the normal lines to the curve $y = \frac{4x - x^2}{4}$ which pass through the point $x_0 = 2$.

Solution. The ordinate of the point $x_0 = 2$ is $y_0 = \frac{4 \cdot 2 - 2^2}{4} = \frac{8 - 4}{4} = 1$. The coordinates of the tangency point are $(2; 1)$. The derivative of the function is $y' = \frac{(4x - x^2)'}{4} = \frac{1}{4}(4 - 2x) = 1 - \frac{x}{2}$. Then $y'(x_0) = y'(2) = 1 - \frac{2}{2} = 0$. Hence $y - 1 = 0(x - 2) \Rightarrow y - 1 = 0 \Rightarrow y = 1$ is the equation of the tangent line and $x = 2$ is the equation of the normal line.

Example 3. Find the equations of the tangent and the normal lines to the curve $4x^4 + 6xy - y^4 = 0$ which pass through the point $M(1; 2)$.

Solution. Substitute the coordinates of the point M into the equation of the curve $4 \cdot 1^4 + 6 \cdot 1 \cdot 2 - 2^4 = 0 \Rightarrow 0 = 0$. The point M belongs to the given curve and $x_0 = 1; y_0 = 2$. The derivative of the implicit function is

$$\begin{aligned} (4x^4 + 6xy - y^4)' &= 0 \Rightarrow 4 \cdot 4x^3 + 6(y + xy') - 4y^3y' = 0 \Rightarrow \\ \Rightarrow 16x^3 + 6y + 6xy' - 4y^3y' &= 0 \Rightarrow 16x^3 + 6y + y'(6x - 4y^3) = 0 \Rightarrow \\ \Rightarrow y' &= \frac{16x^3 + 6y}{4y^3 - 6x} \Rightarrow y'(M) = \frac{16 \cdot 1^3 + 6 \cdot 2}{4 \cdot 2^3 - 6 \cdot 1} = \frac{16 + 12}{32 - 6} = \frac{28}{26} = \frac{14}{13}. \end{aligned}$$

Using (28) and (29) we obtain $y - 2 = \frac{14}{13}(x - 1) \Rightarrow 13y - 26 = 14x - 14 \Rightarrow$

$$\Rightarrow 14x - 13y = -12 \text{ is the equation of the tangent and } y - 2 = -\frac{1}{14/13}(x - 1) \Rightarrow$$

$\Rightarrow y - 2 = -\frac{13}{14}(x - 1) \Rightarrow 14y - 28 = -13x + 13 \Rightarrow 13x + 14y = 41$ is the equation of the normal line.

Example 4. Find the equations of the tangent and the normal lines to the curve $x = \cos^3 t; y = \sin^3 t$ which pass through the point $t = \pi / 4$.

Solution. We have a parametric curve equation. Find

$$x_0 = \cos^3\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}; \quad y_0 = \sin^3\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{4}.$$

Using (23), $y' = \frac{(\sin^3 t)'}{(\cos^3 t)'} = \frac{3\sin^2 t \cos t}{3\cos^2 t (-\sin t)} = -tg t$. $y'\left(\frac{\pi}{4}\right) = -tg \frac{\pi}{4} = -1$.

The equation of the tangent: $y - \frac{\sqrt{2}}{4} = -1(x - \frac{\sqrt{2}}{4}); \quad y - \frac{\sqrt{2}}{4} = -x + \frac{\sqrt{2}}{4};$
 $y = \frac{\sqrt{2}}{2} - x.$

Using (29) the equation of the normal line can be obtained:

$$y - \frac{\sqrt{2}}{4} = -\frac{1}{-1}(x - \frac{\sqrt{2}}{4}) ; \quad y - \frac{\sqrt{2}}{4} = x - \frac{\sqrt{2}}{4}; \quad y = x.$$

Example 5. Find the angle between two curves: $x^2 + y^2 = 8$ and $y^2 = 2x$.

Solution. The angle between the two curves is the angle between their tangents at the intersection point M_0 . Determine the angle following the known formula $tg\phi = \frac{y'_2(x_0) - y'_1(x_0)}{1 + y'_1(x_0)y'_2(x_0)}$. Let's find the intersection point of the given curves

$$\begin{cases} x^2 + y^2 = 8 \\ y^2 = 2x \end{cases} \Rightarrow \begin{cases} x^2 + 2x = 8 \\ y^2 = 2x \end{cases} \Rightarrow \begin{cases} x^2 + 2x - 8 = 0, \\ y^2 = 2x. \end{cases}$$

The first equation has roots $x_1 = -4$ and $x_2 = 2$. Hence $y_{1,2} = \pm\sqrt{2 \cdot 2} = \pm 2$. We obtain two intersection points $M_1(2; 2)$ and $M_2(2; -2)$. Find the angle between the given curves at the point M_1 . The derivative of the function $x^2 + y^2 = 8$ is $2x + 2yy' = 0 \Rightarrow y' = -\frac{2x}{2y} = -\frac{x}{y}; \quad y'(M_1) = -\frac{2}{2} = -1 \quad y'_1(M_1) = -\frac{2}{2} = -1$. The derivative of the function $y^2 = 2x$ is $2yy' = 2 \Rightarrow y' = \frac{2}{2y} = \frac{1}{y}; y'(M_1) = \frac{1}{2}$.

$$\text{Then } tg\phi = \frac{\frac{1}{2} - (-1)}{1 + \frac{1}{2}(-1)} = \frac{\frac{1}{2} + 1}{1 - \frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3; \quad \phi = arctg 3.$$

It is recommended to find the angle between the curves at the point M_2 individually.

Example 6. The law of motion of a material point along a straight line is determined by the formula $x = \frac{t^4}{4} - 4t^3 + 18t^2$. At what point of time, its acceleration equals zero?

Solution. The velocity of the material point $v = x' = (\frac{t^4}{4} - 4t^3 + 18t^2)' = \frac{4t^3}{4} - 4 \cdot 3t^2 + 18 \cdot 2t = t^3 - 12t^2 + 36t$. Then the acceleration of the material point $a = v' = (t^3 - 12t^2 + 36t)' = 3t^2 - 12 \cdot 2t + 36 = 3t^2 - 24t + 36$. If the acceleration equals zero, $3t^2 - 24t + 36 = 0 \Rightarrow t^2 - 8t + 12 = 0 \Rightarrow t_{1,2} = 4 \pm \sqrt{16 - 12} = 4 \pm 2 \Rightarrow t_1 = 6; t_2 = 2$. The acceleration equals zero when $t = 6$ s and $t = 2$ s.

Example 7. A body with a mass of 5 kg moves rectilinearly according to the law $S(t) = t^2 3^t$. Find the kinetic energy of the body after 3 seconds from the start of the movement.

Solution. The kinetic energy is $E = \frac{mv^2}{2}$. Find the velocity $v = S'(t) = (t^2 3^t)' = 2t 3^t + t^2 3^t \ln 3$. Then $v(3) = 2 \cdot 3 \cdot 3^3 + 3^2 \cdot 3^3 \ln 3 = 81(2 + 3 \ln 3)$;
 $E = \frac{5 \cdot 81^2 (2 + 3 \ln 3)^2}{2} = 16402,5(2 + 3 \ln 3)^2$ (J).

Individual Task 6

Solve problems using (28), (29), (30).

Variant 1

1. On the function graph $y = x(x - 4)^3$, find the points where the tangent is parallel to the Ox axis.

2. Find the equations of the tangent and the normal lines to the curve $y = \arcsin \frac{1}{x}$ which pass through the point $x_0 = 2$.

3. The law of a material point is determined by the formula $S = \frac{4t + 3}{t + 4}$. Find the velocity at time $t = 9$ s.

Variant 2

1. Find the points in which the tangents to the curve $y = \frac{x^3}{3} - x^2 - x + 1$ are parallel to the line $y = 2x - 1$.

2. Find the equations of the tangent and the normal lines to the curve $y = \sqrt{x^2 - 9}$ which pass through the point $x_0 = 5$.

3. The law of the motion of two material points along a straight line is determined by the equations $S_1 = 4t^2 + 2$, $S_2 = 3t^2 + 4t - 1$. Find the velocity of the points at a time when the distances covered by them are equal.

Variant 3

1. Find the points in which the tangent to the graph of the function $y = \frac{x + 2}{x - 2}$ forms an angle 135° with the axis Ox.

2. Find the equations of the tangent and the normal lines to the curve $y = \frac{x - 1}{x^2 + 1}$ which pass through the point $x_0 = 0$.

3. A material point of a mass $m = 5$ moves rectilinearly according to the law $S(t) = t \sin^2 3t$. Find the momentum of this material point using the formula $p = mv$ at a time $t = \pi / 6$.

Variant 4

1. Find the points at which the tangent to the graph of the function $y = \frac{x+2}{x-2}$ forms an angle 45° with the axis Ox .
2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = t - \sin t, \\ y = t - \cos t \end{cases}$ which pass through the point $t_0 = \pi / 2$.
3. A material point of mass $m=1$ moves rectilinearly according to the law $S(t) = \frac{t^2 + 3}{\sqrt{t}}$ under the action of the force F . Find the value of force F at time $t=1$ s.

Variant 5

1. Find the points in which the tangents to the curve $y^2 = x$ are parallel to the line $y = 2x + 1$.
2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = 2\cos t - \cos 2t, \\ y = 2\sin t - \sin 2t \end{cases}$ which pass through the point $t_0 = \pi / 2$.
3. The law of a material point motion is determined by the formula $S = \frac{2t+1}{t+2}$. Find the velocity of the motion at time $t = 2$ c.

Variant 6

1. On the graph of the function $y = x(x-1)^2$ find the points at which the tangent parallel abscissa axis.
2. Find the equations of the tangent and the normal lines to the curve $y = \frac{x+2}{x-2}$ which pass through the point $x_0 = 1$.
3. The law of motion of two material points along a straight line is determined by the equations $S_1 = 4t^2 + 2$, $S_2 = 3t^2 + 4t - 1$. Find the point of time when the velocities of the points are equal to each other.

Variant 7

1. Find the points in which the tangents to the curve $y = \frac{x^3}{3} - x^2 + x + 17$ are perpendicular to the line $y = 2x - 5$.
2. Find the equations of the tangent and the normal lines to the curve $y = \sqrt{x^2 - 9}$ which pass through the point $x_0 = 5$.
3. A material point of mass $m=2$ moves rectilinearly according to the law $S(t) = t \cos^2 2t$. Find the momentum of this material point by the formula $p = mv$ at a time $t = \pi / 6$.

Variant 8

1. Find the points in which the tangent to the graph of the function $y = \frac{x+2}{x-2}$ forms an angle 135° with the axis Ox .
2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = 1 - t^2, \\ y = t - t^3 \end{cases}$ which pass through the point $t_0 = 2$
3. A material point of mass $m = 3$ moves rectilinearly according to the law $S(t) = \frac{\sqrt{2t^2 + 5}}{t + 1}$ under the action of the force F . Find the value of force F at time $t = 1$ s.

Variant 9

1. On the graph of the function $y = \frac{x^3}{x-2}$ find the points at which the tangent parallel ordinate axis.
2. Find the equations of the tangent and the normal to the curve $3x - x^2 + y + 2y^2 = 0$ which pass through the point $M(3;0)$.
3. The law of a material point motion is determined by the formula $S = \frac{2t^2 + 1}{\sqrt{t + 2}}$. Find the velocity at time $t = 1$ c.

Variant 10

1. Find the points in which the tangents to the curve $y = \frac{x^3}{3} - x^2 + 3x + 1$ are parallel to the line $y = 6x - 1$.
2. Find the equations of the tangent and the normal lines to the curve $x^2 + 3y - 2x^3y = 1$ which pass through the point $M(1;0)$.
3. The law of a material point motion is determined by the formula $S = \frac{4t + 3}{\sqrt{t + 4}}$. Find the velocity at time $t = 5$ c.

Variant 11

1. Find the angle between the curves $y = x^2$ and $y^2 = x$ at the intersection points.
2. Find the equations of the tangent and the normal lines to the curve $2x^2 + 8x + y^2 + 2 = 0$ which pass through the point $M(-3;2)$.
3. The law of the motion of two material points along a straight line is determined by the equations $S_1 = 4t^2 + 3$, $S_2 = 3t^2 + 2t - 1$. Find the velocity of the points at a time when the distances covered by them are equal to each other.

Variant 12

1. Find the point of the curve $y = 3x^2 - 5x + 1$ at which the slope of the normal line to this curve equals to 2.

2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = \sqrt{5+t^2}, \\ y = 2t + t^3 \end{cases}$ which pass through the point $t_0 = 2$.

3. The law of a material point motion is determined by the formula $S = \frac{2t+3}{t+5}$.

Find the velocity at $t = 4$ c.

Variant 13

1. Find the point at which the tangent to the curve $y = x^5 - 5x^4 + 20x^2 - 3$ is perpendicular to the line $y = -\frac{1}{160}x + 7$ (all cases).

2. Find the equations of the tangent and the normal lines to the curve $3x^2 + 7x + y^2 + 5 = 0$ which pass through the point $M(-1;2)$.

3. The law of the motion of two material points along a straight line is determined by the equations $S_1 = 3t^2 + 2t + 6$, $S_2 = 2t^2 + 5t - 1$. Find the point of time when the velocities of the points are equal to each other.

Variant 14

1. Find the equations of tangent and normal lines to the curve $y = \frac{10}{x^2+1}$ at the points of its intersection with the parabola $y = \frac{x^2}{2}$.

2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = 4\sin^3 t, \\ y = 4\cos^3 t \end{cases}$ which pass through the point $t_0 = \pi/4$.

3. A material point of a mass $m = 4$ moves rectilinearly according to the law $S(t) = \sin^2 2t$. Find the momentum of this material point using the formula $p = mv$ at a time $t = \pi/3$.

Variant 15

1. The line $y = \frac{5}{6}x + \frac{7}{12}$ is parallel to the tangent to the graph of the function $y = 2x^3 - 3x^2 - \frac{67}{6}x$. Find the tangency point coordinates (all cases).

2. Find the equations of the tangent and the normal lines to the curve $\begin{cases} x = 2\cos t, \\ y = 2\sin 2t \end{cases}$ which pass through the point $t_0 = \pi/2$.

3. A material point of a mass $m = 3$ moves rectilinearly according to the law $S(t) = \frac{t^2 + 3}{t}$ under the action of the force F . Find the value of force F at time $t = 1$ s.

Variant 16

1. The point with ordinate $y = -2$ lies on the curve $y = x^3 - 2x^2 - x$. Find the equations of the tangent and the normal lines to the curve at this point (all cases).

2. Find the equations of the tangent and the normal lines to the curve $y = \frac{x-1}{x^2+1}$ which pass through the point $x_0 = 0$.

3. The law of a material point motion is determined by the formula $S = \frac{2t^2 + 3}{t + 2}$. Find the velocity at time $t = 2$ c.

Variant 17

1. The secant of the parabola $y = 3x^2 + 6x - 2$ connects points with abscissas $x_1 = -1$ and $x_2 = 1$. Find an equation of the tangent which is parallel to the secant.

2. Find the equations of the tangent and the normal lines to the curve $y = \sqrt{x^2 - 5}$ which pass through the point $x_0 = 3$.

3. The law of motion of two material points along a straight line is determined by the equations $S_1 = 6t^2 + 2t + 3$, $S_2 = 3t^2 + 4t - 1$. Find the point of time when the velocities of the points are equal to each other.

Variant 18

1. Find the angle between the curve $y = 3 \cdot 2^{-x}$ and the straight line $y = 3$.

2. On the graph of the function $y = \frac{x^3}{x-4}$ find the points at which the tangent line parallel to the abscissa axis.

3. A material point of mass $m = 5$ moves rectilinearly according to the law $S(t) = \frac{t+2}{2t^2+1}$ under the action of the force F . Find the value of force F at time $t = 1$ s.

Variant 19

1. On the graph of the function $y = \frac{x^3}{x+5}$ find the points at which the tangent parallel to the abscissa axis.

2. Find the equations of the tangent and the normal lines to the curve $y = \arcsin \frac{1}{x}$ which pass through the point $x_0 = 2$.

3. The law of a material point motion is determined by the formula $S = \frac{5t - 3}{3t + 4}$.

Find the velocity at time $t = 3$ c.

Variant 20

1. Find the points at which the tangent to the graph of the function $y = \frac{x + 2}{x - 2}$

forms an angle 45° with the axis Ox .

2. Find the equations of the tangent and the normal lines to the curve

$$\begin{cases} x = t - 2t^2, \\ y = 3t - t^3 \end{cases} \quad \text{which pass through the point } t_0 = 1.$$

3. A material point of mass $m = 3$ moves rectilinearly according to the law

$S(t) = \frac{t^2 - 1}{t + 1}$ under the action of the force F . Find the value of force F at time $t = 1$ s.

2. Differential

2.1. Definition and Geometric Meaning of the Differential

We have defined the derivative or differential coefficient as $y' = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$,

where dy/dx was not to be regarded as dy divided by dx but as the limit of the quotient $\Delta y/\Delta x$ as $\Delta x \rightarrow 0$.

There are, however, situations where it is important to give separate meanings to dx and dy .

Let the function $y = f(x)$ is differentiable at the point x , then $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$.

Hence $\frac{\Delta y}{\Delta x} = f'(x) + \alpha$, where $\alpha \rightarrow 0$ as $\Delta x \rightarrow 0$.

Then $\Delta y = f'(x)\Delta x + \alpha\Delta x$. The first term in the right-hand side of this equality is linear with respect to Δx and the second term is an infinitesimal of a higher order than Δx . The first term is the main part of the increment of the function. It is referred to as the differential of the function.

It is clear that $\Delta x = dx$, so dx is called the differential of x . Hence

$$dy = f'(x)dx \quad (31)$$

The differential of the function $f(x)$ for given values of x and Δx equals to the increment QN of the ordinate of the tangent MQ to the curve $y = f(x)$ at the point M , when the argument of the increment is Δx (Fig. 2).

This is the geometric meaning of the differential.

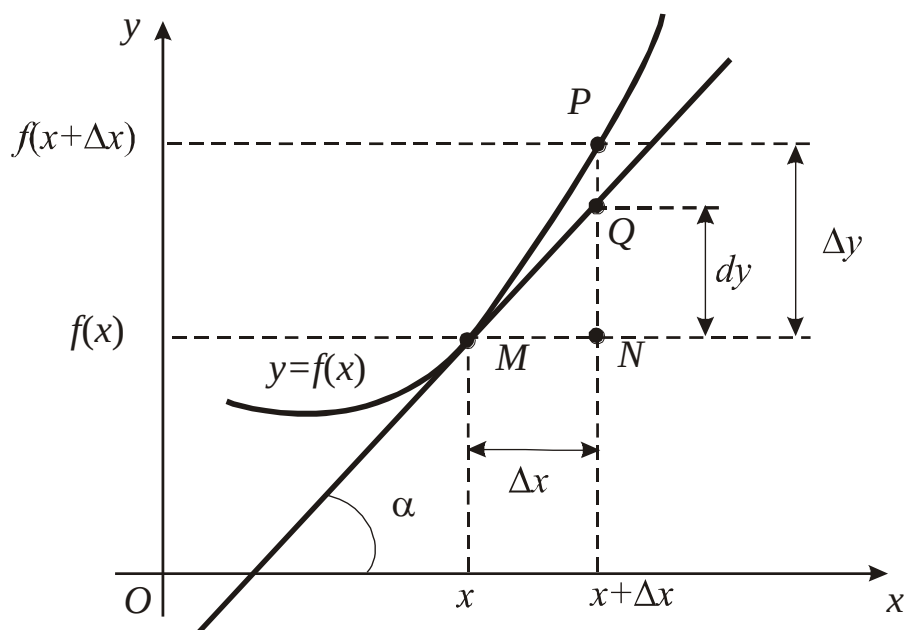


Fig. 2

2.2. The Main Properties of the Differential

Let $u(x)$, $v(x)$ are differentiable functions. Then

1. $dc = 0$ ($c - \text{const}$)
2. $d(u \pm v) = du \pm dv$
3. $d(uv) = vdu + u dv$
4. $dcu = cdu$ ($c - \text{const}$)
5. $d\left(\frac{u}{v}\right) = \frac{vdu - u dv}{v^2}$, $v \neq 0$
6. $df(u) = f' du$, $u = u(x)$.

The last equation is called the invariance property of the first-order differential form

2.3. Applications of the Differential $dy = f'(x)dx$ to Approximate Calculations and Error Theory

According to fig. 2 the differential of the function dy is approximately equal to Δy for the small increment Δx i.e. $\Delta y \approx dy$, therefore

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x. \quad (32)$$

The relative error when calculating the value of the function y can be approximately determined using the differential, i.e.

$$\left| \frac{\Delta y}{y} \right| \approx \left| \frac{dy}{y} \right|. \quad (33)$$

Example 1. Find the differential of the function $y = \arctg \sqrt{x^2 - 1} + \ln \cos x^2$.

Solution. Using (31) we have

$$\begin{aligned} dy &= d(\arctg \sqrt{x^2 - 1} + \ln \cos x^2) = (\arctg \sqrt{x^2 - 1} + \ln \cos x^2)' dx = \\ &= \left(\frac{1}{(\sqrt{x^2 - 1})^2 + 1} \cdot \frac{1}{2\sqrt{x^2 - 1}} 2x + \frac{1}{\cos x^2} (-\sin x^2) 2x \right) dx = \\ &= \left(\frac{x}{x^2 \sqrt{x^2 - 1}} - 2x \operatorname{tg} x^2 \right) dx = \left(\frac{1}{x \sqrt{x^2 - 1}} - 2x \operatorname{tg} x^2 \right) dx. \end{aligned}$$

Example 2. Find the differential of the function $x^3 + y^3 + 3xy - 15 = 0$ at point $M(1; 2)$.

Solution. Find the derivative of the implicit function

$$\begin{aligned} 3x^2 + 3y^2 y' + 3y + 3xy' &= 0 \Rightarrow x^2 + y^2 y' + y + xy' = 0 \Rightarrow y'(y^2 + x) = -(x^2 + y). \\ \Rightarrow y' &= -\frac{x^2 + y}{y^2 + x}. \text{ Using (31) we have } dy(M) = -\frac{1^2 + 2}{2^2 + 1} dx = -\frac{3}{5} dx. \end{aligned}$$

Example 3. Find the approximate value (Evaluate) $\sqrt[3]{30}$.

Solution. Let us consider the function $f(x) = \sqrt[3]{x}$. Using (32) we obtain $\sqrt[3]{x + \Delta x} \approx \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}} \Delta x$. In the given case we have $x + \Delta x = 30$. For $x = 27$ we obtain $\Delta x = 3$ and $\sqrt[3]{27 + 3} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}} 3 = 3 + \frac{3}{3 \cdot 9} = 3 + \frac{1}{9} = \frac{28}{9}$ then $\sqrt[3]{30} \approx \frac{28}{9}$.

Example 4. Evaluate $\cos 155^\circ$.

Solution. Let us consider the function $f(x) = \cos x$. Using (32), $\cos(x + \Delta x) \approx \cos x + (\cos x)' \Delta x = \cos x - \sin x \Delta x$.

Convert degrees into radians $155^\circ = \frac{\pi}{180} 155^\circ = \frac{31\pi}{36}$. Let $x = 150^\circ = \frac{5\pi}{6}$;
 $x + \Delta x = \frac{31\pi}{36}$, i.e. $\frac{5\pi}{6} + \Delta x = \frac{31\pi}{36}$; $\Delta x = \frac{31\pi}{36} - \frac{5\pi}{6} = \frac{\pi}{36}$. We obtain
 $\cos 155^\circ = \cos \frac{31\pi}{36} \approx \cos \frac{5\pi}{6} - \sin \frac{5\pi}{6} \cdot \frac{\pi}{36}$. Using trigonometric formulas,
 $\cos \frac{5\pi}{6} = \cos(\pi - \frac{\pi}{6}) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$; $\sin \frac{5\pi}{6} = \sin(\pi - \frac{\pi}{6}) = \sin \frac{\pi}{6} = \frac{1}{2}$.

Then

$$\cos 155^\circ = \cos \frac{31\pi}{36} \approx \cos \frac{5\pi}{6} - \sin \frac{5\pi}{6} \cdot \frac{\pi}{36} = -\frac{\sqrt{3}}{2} - \frac{1}{2} \frac{\pi}{36} = -\frac{\sqrt{3}}{2} - \frac{\pi}{72} \approx -0,908.$$

Example 5. Find the relative error in calculating the volume of the sphere, if the error in determining its radius was Δr .

Solution. Using (33) $\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{V'(r)\Delta r}{V(r)} = \frac{\left(\frac{4}{3}\pi r^3\right)' \Delta r}{\frac{4}{3}\pi r^3} = \frac{4\pi r^2 \Delta r}{\frac{4}{3}\pi r^3} = 3 \frac{\Delta r}{r}.$

So, the relative error when determining the volume of the sphere is approximately equal to three times the relative error that was made when calculating the radius of the sphere.

Individual Task 7

Solve problems using (31), (32), (33).

Variant 1

1. Find the differential of the function $y = 3x^2 - x$ at the point $x = 1$.
2. Using the differential, find the approximate value of $\cos 9^\circ$.
3. Find the differential of the function $y = \frac{\cos x}{1 - x^2}$

Variant 2

1. Find the differential of the function $y = \frac{5}{x^2} + \arctg \frac{x}{5}$ at the point $x = 1$.
2. Using the differential, find the approximate value of $\sqrt{26}$.
3. Find the differential of the function $y = tg^2 x$

Variant 3

1. Find the differential of the function $y = \frac{\sqrt[3]{x}}{0.2}$.
2. Using the differential, find the approximate value of $\ln 1,1$.
3. Find the differential of the function $y = 5^{\ln tg x}$

Variant 4

1. Find the differential of the function $y = \frac{x^3 + 1}{x^3 - 1}$.
2. Using the differential, find the approximate value of $\sqrt[4]{83}$
3. Find the differential of the function $y = \sqrt{\arcsin x} + (\arctg x)^2$

Variant 5

1. Find the differential of the function $y = 2^{-1/\cos x}$.
2. Using the differential, find the approximate value of $\arcsin 0,02^\circ$
3. Find the differential of the function $y = 3^{-1/x^2} + 3x^2 - 4\sqrt{x}$

Variant 6

1. Find the differential of the function $y = \ln tg\left(\frac{\pi}{2} - \frac{x}{4}\right)$.
2. Using the differential, find the approximate value of $\sqrt{27}$

3. Find the differential of the function $y = \sqrt{\cos 5x^2} + 3x^2\sqrt{x}$

Variant 7

1. Find the differential of the function $y = \ln \operatorname{tg}\left(\frac{\pi}{2} - \frac{x}{4}\right)$.

2. Using the differential, find the approximate value of $\sin 12^\circ$

3. Find the differential of the function $y = \ln \operatorname{tg}\left(\frac{\pi}{2} - \frac{x}{4}\right)$

Variant 8

1. Find the differential of the function $y = \frac{\cos x}{1 - x^2}$.

2. Using the differential, find the approximate value of $\operatorname{tg} 12^\circ$.

3. Find the differential of the function $y = 2^{-1/\cos x}$

Variant 9

1. Find the differential of the function $y = \operatorname{tg}^2 x$.

2. Using the differential, find the approximate value of $\ln 1,2$.

3. Find the differential of the function $y = \frac{x^3 + 1}{x^3 - 1}$

Variant 10

1. Find the differential of the function $y = 5^{\ln \operatorname{tg} x}$.

2. Using the differential, find the approximate value of $\arccos 0,02^\circ$.

3. Find the differential of the function $y = \frac{\sqrt[3]{x}}{0.2}$

Variant 11

1. Find the differential of the function $y = \sqrt{\arcsin x} + (\operatorname{arctg} x)^2$.

2. Using the differential, find the approximate value of $\sin 9^\circ$.

3. Find the differential of the function $y = \frac{5}{x^2} + \operatorname{arctg} \frac{x}{5}$ at the point $x = 1$.

Variant 12

1. Find the differential of the function $y = 3^{-1/x^2} + 3x^2 - 4\sqrt{x}$.

2. Using the differential, find the approximate value (evaluate) of $\sqrt[3]{131}$.

3. Find the increment and the differential of the function $y = 3x^2 - x$ upon transition of an independent variable from value $x = 1$ to value $x = 1,02$.

Variant 13

1. Find the increment and the differential of the function $y = 2x^3 - x^2$ upon transition of an independent variable from value $x = 1$ to value $x = 1,02$.

2. Using the differential, find the approximate value of $\operatorname{ctg} 9^\circ$.

3. Find the differential of the function $y = \frac{\sqrt{\cos x}}{1 - x^2}$

Variant 14

1. Find the differential of the function $y = \frac{3}{x^2} - \operatorname{arccctg} \frac{x}{3}$ at the point $x = 1$.
2. Using the differential, find the approximate value of $\sqrt{17}$.
3. Find the differential of the function $y = \operatorname{ctg}^2 \sqrt{x}$

Variant 15

1. Find the differential of the function $y = \frac{\sqrt[3]{x+1}}{0.2x}$.
2. Using the differential, find the approximate value of $\ln 1,3$.
3. Find the differential of the function $y = 5^{\ln \cos x}$

Variant 16

1. Find the differential of the function $y = \frac{2x^3 + 1}{\sqrt{x^3 - 1}}$.
2. Using the differential, find the approximate value of $\sqrt[4]{630}$
3. Find the differential of the function $y = \sqrt{\arccos x} + (\operatorname{arccctg} x)^2$

Variant 17

1. Find the differential of the function $y = 5^{\frac{1}{\sin x}}$.
2. Using the differential, find the approximate value of $\arcsin 0,1^\circ$.
3. Find the differential of the function $y = 3^{-1/x} + 3x^2 - \frac{4}{\sqrt{x}}$

Variant 18

1. Find the differential of the function $y = \ln \cos\left(\frac{\pi}{2} - \frac{x}{4}\right)$.
2. Using the differential, find the approximate value of $\sqrt{10}$
3. Find the differential of the function $y = \sqrt{\operatorname{tg} x^2} + 3x^2$

Variant 19

1. Find the differential of the function $y = \sqrt[3]{\operatorname{tg}\left(\frac{\pi}{2} - \frac{x}{4}\right)}$.
2. Using the differential, find the approximate value of $\operatorname{ctg} 12^\circ$
3. Find the differential of the function $y = \cos \ln \frac{x}{4}$

Variant 20

1. Find the differential of the function $y = \frac{\operatorname{tg} x}{1 - x^2}$.
2. Using the differential, find the approximate value of $\operatorname{ctg} 31^\circ$.
3. Find the differential of the function $y = 5^{\frac{1}{\ln x}}$

3. Higher Derivatives and Differentials

3.1. Higher Derivatives

Let a differentiable function $y = f(x)$ is given in the interval (a,b) . We can differentiate the derivative one more time with respect to x . If the function $f'(x)$ also has a derivative in the interval (a,b) , then this derivative is called the second derivative and is denoted by one of the symbols y'' ; $f''(x)$; $\frac{d^2y}{dx^2}$; $\frac{d^2f}{dx^2}$.

If the derivative from the second derivative exists, it is called a third-order derivative, i.e. $y''' = \frac{d}{dx}(\frac{d^2y}{dx^2})$.

By repeated differentiation, we can obtain the 4th, 5th, ..., nth derivative.

In general, $y^{(n)} = (y^{(n-1)})'$.

The derivatives above the first order are called higher derivatives. Derivatives above the third order are indicated by numbers in brackets.

Example 1. Find the third-order derivative of the function $y = x^5 - 3x^4 + 2x^3 - 3$.

Solution. Consistently find: $y' = 5x^4 - 12x^3 + 6x^2$; $y'' = 20x^3 - 36x^2 + 12x$; $y''' = 60x^2 - 72x + 12$.

Example 2. Find the n - order derivative of the function $y = a^x$.

Solution. $y' = a^x \ln a$; $y'' = a^x \ln a \cdot \ln a = a^x \ln^2 a$; $y''' = a^x \ln a \cdot \ln a \cdot \ln a = a^x \ln^3 a$; ... ; $y^{(n)} = a^x \ln^n a$.

3.2. Calculation of the Second-order Derivatives of the Parametric Functions

The second-order derivative ($\frac{d^2y}{dx^2}$ or y''_{xx}) of the parametric function $x = x(t)$, $y = y(t)$ is

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'_t}{x'_t} \quad (34)$$

or

$$\frac{d^2y}{dx^2} = \frac{y''_t x'_t - y'_t x''_t}{(x'_t)^3}. \quad (35)$$

Example 1. Find $\frac{d^2y}{dx^2}$ of the parametric function $\begin{cases} x = \ln t, \\ y = t^2 - 1 \end{cases}$.

Solution. We have $\frac{dy}{dx} = \frac{(t^2 - 1)'_t}{(\ln t)'_t} = \frac{2t}{\frac{1}{t}} = 2t^2$. Using (34)

$$\frac{d^2y}{dx^2} = \frac{(2t^2)'_t}{(\ln t)'_t} = \frac{4t}{\frac{1}{t}} = 4t^2.$$

Example 2. Find $\frac{d^2y}{dx^2}$ of the parametric function $\begin{cases} x = 2\cos t, \\ y = 3\sin t \end{cases}$.

Solution. Using (35), we find the second derivatives: $x'_t = -2\sin t$, $x''_{tt} = -2\cos t$, $y'_t = 3\cos t$, $y''_{tt} = -3\sin t$. Using (35),

$$\frac{d^2y}{dx^2} = \frac{-3\sin t(-2\sin t) - (-2\cos t)3\cos t}{(-2\sin t)^3} = \frac{6\sin^2 t + 6\cos^2 t}{-8\sin^3 t} = -\frac{3}{4\sin^3 t}.$$

3.3. Higher-Order Derivatives of the Implicit Functions

Пусть функция задана неявно $F(x, y) = 0$. Чтобы найти первую производную продифференцируем это равенство по x и решим полученное уравнение относительно y' . To find the second derivative of the implicit function we can differentiate the first derivative with respect to x and substitute its value in the resulting formula. If we continue differentiation, we can find derivatives of any order one after another. All of them will be expressed in terms of the independent variable x and the function y .

Example 1. Find y'' of $y = \sin(x + y)$.

Solution. We differentiate the given function with respect to x $y' = \cos(x + y) \cdot (1 + y')$ and find y' explicitly:

$$y' = \cos(x + y) + y' \cos(x + y) \Rightarrow y'(1 - \cos(x + y)) = \cos(x + y) \Rightarrow$$

$$\Rightarrow y' = \frac{\cos(x + y)}{1 - \cos(x + y)}.$$

We differentiate the resulting ratio with respect to x :

$$y'' = \frac{-\sin(x + y)(1 + y')(1 - \cos(x + y)) - \cos(x + y)\sin(x + y)(1 + y')}{(1 - \cos(x + y))^2};$$

$$y'' = \frac{(1 + y')(-\sin(x + y) + \sin(x + y)\cos(x + y)) - \sin(x + y)\cos(x + y)}{(1 - \cos(x + y))^2} =$$

$$= \frac{(1 + y')(-\sin(x + y))}{(1 - \cos(x + y))^2}.$$

$$\text{Meaning } y' = \frac{\cos(x+y)}{1-\cos(x+y)}, \text{ we have } y'' = -\frac{\left(1 + \frac{\cos(x+y)}{1-\cos(x+y)}\right) \sin(x+y)}{(1-\cos(x+y))^2} =$$

$$= -\frac{(1-\cos(x+y) + \cos(x+y)) \sin(x+y)}{(1-\cos(x+y))^3} = -\frac{\sin(x+y)}{(1-\cos(x+y))^3}.$$

3.4. Higher Differentials

Let the function $y = f(x)$ has all derivatives from 1st to n th order. The n th-order differential is a differential of the $(n-1)$ th-order differential of that function:

$$d^n y = d(d^{n-1} y), \text{ i.e. } d^2 y = f''(x) dx^2; \quad d^3 y = f'''(x) dx^3 \dots$$

Example 1. Find $d^2 y$ of $y = \sqrt[3]{x^2}$.

Solution. We obtain $y' = \left(\sqrt[3]{x^2}\right)' = \frac{2}{3} x^{-1/3}; \quad y'' = \frac{2}{3} \left(-\frac{1}{3}\right) x^{-4/3} = -\frac{2}{9\sqrt[3]{x^4}}.$

$$\text{Then } d^2 y = -\frac{2}{9\sqrt[3]{x^4}} dx^2 = -\frac{2}{9x\sqrt[3]{x}} dx^2.$$

Example 2. Find $d^3 y$, of $y = \sin^2 x$.

Solution. We obtain $y'' = 2 \sin x \cos x = \sin 2x; \quad y'' = 2 \cos 2x;$
 $y''' = -4 \sin 2x.$ Hence $d^3 y = -4 \sin 2x dx^3.$

Individual Task 8

Using (34), (35), find the second-order derivatives of given functions.

Variant 1

$$1. y = x e^{x^2}$$

$$2. x^3 + y^3 = 3xy$$

$$3. \begin{cases} x = \arcsin t, \\ y = \ln(1-t^2); \end{cases}$$

Variant 2

$$1. y = \frac{1}{1+x^3}$$

$$2. \ln(x+y) = y-x$$

$$3. \begin{cases} x = at \cos t, \\ y = at \sin t; \end{cases}$$

Variant 3

$$1. y = (1+x^2) \arctg x$$

$$2. e^{x+y} = xy$$

$$3. \begin{cases} x = e^t \sin t, \\ y = e^t \cos t; \end{cases}$$

Variant 4

$$1. y = \sqrt{(4-x^2)}$$

$$2. e^y + 4xy = y^2$$

$$3. \begin{cases} x = \arctg t, \\ y = t^2 / 2. \end{cases}$$

Variant 5

$$1. y = tg(\sqrt{1+x^2})$$

$$2. tg(xy) = y-x$$

$$3. \begin{cases} x = 3\sqrt{\cos 3t}, \\ y = 2t^3 \sin t^2; \end{cases}$$

Variant 6

$$1. y = 4\cos^3 x - \cos x$$

$$2. e^{5x+y} = \sin xy$$

$$3. \begin{cases} x = e^{\sqrt{t}} \sin t, \\ y = e^{\sqrt{t}} \sqrt{\cos 2t}; \end{cases}$$

Variant 7

$$1. y = \sqrt{\sin x} \cdot \ln^2 x$$

$$2. x^3 + 2xy^3 = 3x\sqrt{y}$$

$$3. \begin{cases} x = \arcsin t, \\ y = \sqrt{(1-t^2)}; \end{cases}$$

Variant 8

$$1. y = 5ctg \sqrt{1+x^3}$$

$$2. tg(xy) = \sqrt{y-3x}$$

$$3. \begin{cases} x = 5t \cos 2t, \\ y = 5 \sin 2t; \end{cases}$$

Variant 9

$$1. y = (1+x^2) \cos^3 x$$

Variant 10

$$1. x^y + xy = e^x$$

Variant 11

$$1. y = tg(x + \sqrt{1+x^2})$$

Variant 12

$$1. y = 3 \ln^2 x \cdot \sin 3x$$

$$\begin{array}{llll}
2. \ln(2x+y) = xy & 2. y = \frac{2x}{\sqrt{(4-x^2)}} & 2. \cos(xy) = 5y - x & 2. 5\sqrt[3]{xy} = 2\sin xy \\
3. \begin{cases} x = \sin^4 t, \\ y = 5^{3t}; \end{cases} & 3. \begin{cases} x = ctg 2t, \\ y = \sqrt[3]{t^2}. \end{cases} & 3. \begin{cases} x = 5\sqrt{\cos 4t}, \\ y = 5\sin^4 2t; \end{cases} & 3. \begin{cases} x = e^t \sin t, \\ y = 4\sqrt{\cos t}; \end{cases}
\end{array}$$

Variant 13

$$\begin{array}{l}
1. y = e^{\sqrt{x}} \\
2. x^3 y^3 = 3x - \sqrt{y} \\
3. \begin{cases} x = e^t, \\ y = t^2 e^t \end{cases}
\end{array}$$

Variant 14

$$\begin{array}{l}
1. y = \sqrt{1-x^2} \sin x \\
2. \sin(x-y) = y - x \\
3. \begin{cases} x = 5t^5, \\ y = 5tctgt; \end{cases}
\end{array}$$

Variant 15

$$\begin{array}{l}
1. y = \frac{\sqrt{1-x^2}}{\arcsin x} \\
2. tg(x+4y) = 4xy \\
3. \begin{cases} x = 2e^{2t}, \\ y = e^t \cos 2t; \end{cases}
\end{array}$$

Variant 16

$$\begin{array}{l}
1. y = 5x^3 \cos 3x \\
2. 5ctgx + xy = y \\
3. \begin{cases} x = \arctgt^2, \\ y = 3\sin 5x. \end{cases}
\end{array}$$

Variant 17

$$\begin{array}{l}
1. y = 5^{\sqrt{1+x^2}} \\
2. \ln(xy) = y - x^4 \\
3. \begin{cases} x = 2\sqrt{\cos^3 t}, \\ y = 2\sin \sqrt{t}; \end{cases}
\end{array}$$

Variant 18

$$\begin{array}{l}
1. y = tg \sqrt{x} - \sqrt{\sin x} \\
2. \sqrt[3]{5x+2y} = \sin xy \\
3. \begin{cases} x = e^{2\sqrt{t}} \\ y = \sqrt{\cos 2t^4}; \end{cases}
\end{array}$$

Variant 19

$$\begin{array}{l}
1. y = \sqrt{\ln x} \cdot tg^2 2x \\
2. 3x^2 - 2xy^3 = 5y\sqrt{x} \\
3. \begin{cases} x = 5\arcsin 3t, \\ y = \sqrt{(1-9t^2)}; \end{cases}
\end{array}$$

Variant 20

$$\begin{array}{l}
1. y = 5^{\sqrt{1+x^3}} \\
2. tg\left(\frac{2x}{y}\right) = \sqrt{y^2 - x} \\
3. \begin{cases} x = 3t \cos^2 t, \\ y = 3\sin 2t; \end{cases}
\end{array}$$

4. Lopital's Rule

The Lopital's rule is a tool for finding the limit of a function when revealing the indeterminate such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theorem (Lopital's rule) Let the functions $f(x)$ and $g(x)$:

1) be defined and differentiable in a vicinity of a point x_0 , except perhaps the point x_0 itself, moreover $g'(x) \neq 0$ in this vicinity;

2) $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$ or $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = \infty$;

3) the finite limit $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ exists.

Then the limit $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}. \quad (36)$$

Note that sometimes an error is made to look for a limit $\lim_{x \rightarrow x_0} \left(\frac{f(x)}{g(x)} \right)'$ instead of a limit $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$. L'opital's rule applies to the calculation of basic indeterminates $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞ , 0^0 , ∞^0 can be reduced to basic ones.

Indeterminate $0 \cdot \infty$ ($\lim_{x \rightarrow x_0} f(x)g(x)$ if $\lim_{x \rightarrow x_0} f(x) = 0$ and $\lim_{x \rightarrow x_0} g(x) = \infty$) can be reduced to the basic indeterminates as follows:

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \left\{ \frac{0}{0} \right\} \text{ or } f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}} = \left\{ \frac{\infty}{\infty} \right\}. \quad (37)$$

Indeterminate $\infty - \infty$ ($\lim_{x \rightarrow x_0} f(x) - g(x)$ if $\lim_{x \rightarrow x_0} f(x) = \infty$ and $\lim_{x \rightarrow x_0} g(x) = \infty$) can be reduced to the basic indeterminates as follows:

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}} = \left\{ \frac{0}{0} \right\}. \quad (38)$$

Indeterminate 1^∞ , 0^0 , ∞^0 can be reduced to the indeterminate $0 \cdot \infty$ using preliminary calculation of the logarithm, i.e. representing the function as

$$(f(x))^{g(x)} = e^{g(x) \ln f(x)}. \quad (39)$$

Example 1. Find $\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x}$.

Solution. $\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x} = \left\{ \frac{0}{0} \right\} = \frac{(e^{3x} - e^{2x})'}{x'} = \lim_{x \rightarrow 0} \frac{3e^{3x} - 2e^{2x}}{1} = 3 - 2 = 1.$

Example 2. Find $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$.

Solution. $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \left\{ \frac{\infty}{\infty} \right\} = \lim_{x \rightarrow \infty} \frac{(\ln x)'}{(x^2)'} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0.$

Example 3. Find $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

Solution.

$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \left\{ \frac{0}{0} \right\} = \lim_{x \rightarrow 0} \frac{(e^x - e^{-x} - 2x)'}{(x - \sin x)'} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \left\{ \frac{0}{0} \right\}.$ We reapply

the Lopital's rule: $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} = \left\{ \frac{0}{0} \right\}$. Once again, we apply the Lo-

pital's rule: $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2$.

Example 4. Find $\lim_{x \rightarrow \infty} ((\pi - 2 \arctg x) \ln x)$.

Solution. $\lim_{x \rightarrow \infty} ((\pi - 2 \arctg x) \ln x) = \{0 \cdot \infty\}$. Using (36), we obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} ((\pi - 2 \arctg x) \ln x) &= \lim_{x \rightarrow \infty} \frac{(\pi - 2 \arctg x)}{\frac{1}{\ln x}} = \left\{ \frac{0}{0} \right\} = \lim_{x \rightarrow \infty} \frac{-\frac{2}{1+x^2}}{-\frac{1}{x \ln^2 x}} = \lim_{x \rightarrow \infty} \frac{2x \ln^2 x}{(1+x^2)} = \left\{ \frac{\infty}{\infty} \right\} = \\ &= 2 \lim_{x \rightarrow \infty} \frac{\ln^2 x + x 2 \ln x \frac{1}{x}}{2x} = \lim_{x \rightarrow \infty} \frac{\ln^2 x + 2 \ln x}{x} = \left\{ \frac{\infty}{\infty} \right\} = \lim_{x \rightarrow \infty} \frac{2 \ln x \frac{1}{x} + 2 \frac{1}{x}}{1} = 2 \lim_{x \rightarrow \infty} \frac{\ln x + 2}{x} = \\ &= \left\{ \frac{\infty}{\infty} \right\} = 2 \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 2 \lim_{x \rightarrow \infty} \frac{1}{x} = 0. \end{aligned}$$

Example 5. Find $\lim_{x \rightarrow \frac{\pi}{2}} (tg x - \frac{1}{\cos x})$.

Solution. $\lim_{x \rightarrow \frac{\pi}{2}} (tg x - \frac{1}{\cos x}) = \{\infty - \infty\} = \lim_{x \rightarrow \frac{\pi}{2}} (\frac{\sin x}{\cos x} - \frac{1}{\cos x}) =$
 $= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos x} = \left\{ \frac{0}{0} \right\} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$.

Example 6. Find $\lim_{x \rightarrow 0} x^{\sin x}$.

Solution. $\lim_{x \rightarrow 0} x^{\sin x} = \{0^0\}$. We calculate the logarithm of the function $y = \lim_{x \rightarrow 0} x^{\sin x}$. We obtain $\ln y = \ln \lim_{x \rightarrow 0} x^{\sin x}$. We know that $\ln \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \ln f(x)$. Then

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0} \ln x^{\sin x} = \lim_{x \rightarrow 0} \sin x \ln x = \{0 \cdot \infty\} = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = - \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos x} = \{ \text{as} \\ &\cos 0 = 1 \} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{1} = 0, \text{ i.e. } \ln y = 0 \text{ and } y = e^0 = 1. \end{aligned}$$

Individual Task 9

Using (36–39), find the limits of functions by the L'opital's rule.

Variant 1

1. $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 2}{x^3 - 4x^2 + 3}$
2. $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$
3. $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{tgx}$

Variant 2

1. $\lim_{x \rightarrow 0} \frac{\ln \cos x}{x}$
2. $\lim_{x \rightarrow 0} \frac{e^{tgx} - e^x}{tgx - x}$
3. $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}}$

Variant 3

1. $\lim_{x \rightarrow 1} \frac{\ln(x-1)}{ctg \pi x}$
2. $\lim_{x \rightarrow 0} \frac{\ln \sin 2x}{\ln \sin x}$
3. $\lim_{x \rightarrow 0} \left(\frac{2}{x} \right)^{\sin x}$

Variant 4

1. $\lim_{x \rightarrow 0} \frac{e^{-x} + x - 1}{x^2}$
2. $\lim_{x \rightarrow \frac{\pi}{2}} (tgx)^{2 \cos x}$
3. $\lim_{x \rightarrow 0} (5^x + x)^{\frac{2}{x}}$

Variant 5

1. $\lim_{x \rightarrow \infty} \frac{x^4 - 3x^2 + 2}{2x^3 + 4x^2 + 3}$
2. $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$
3. $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{\sin 2x}$

Variant 6

1. $\lim_{x \rightarrow 0} \frac{\ln \cos x}{2 \sin x}$
2. $\lim_{x \rightarrow 0} \frac{\cos x - e^x}{\sin x - 2x}$
3. $\lim_{x \rightarrow 0} \left(\frac{1-x}{1+x} \right)^{\frac{1}{x}}$

Variant 7

1. $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{\operatorname{arcctg} \left(\frac{1}{x} \right)}$
2. $\lim_{x \rightarrow 0} \frac{\sin 2x}{tgx}$
3. $\lim_{x \rightarrow 0} \left(\frac{x+2}{x} \right)^{\sin x}$

Variant 8

1. $\lim_{x \rightarrow 0} \frac{e^{-x} + \sin x - 1}{3x^2}$
2. $\lim_{x \rightarrow \pi} (\sin x)^{ctgx}$
3. $\lim_{x \rightarrow 0} \left(\frac{5+x}{x-2} \right)^{\frac{2}{x}}$

Variant 9

1. $\lim_{x \rightarrow \infty} \frac{4x^3 + 3x^2 + 2}{x^5 + 4x^2 - 3}$
2. $\lim_{x \rightarrow 1} \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\ln^2 x} \right)$
3. $\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt[3]{x}} \right)^{tg 5x}$

Variant 10

1. $\lim_{x \rightarrow 0} \frac{\ln \cos x}{x^2 + x}$
2. $\lim_{x \rightarrow 0} \frac{4^{tgx} - 4^x}{\sin 3x - x}$
3. $\lim_{x \rightarrow 0} (5^x + x)^{\frac{1}{2x}}$

Variant 11

1. $\lim_{x \rightarrow 1} \ln x \cdot \ln(x-1)$
2. $\lim_{x \rightarrow \infty} (2^{\frac{1}{x}} - 1) \cdot x$
3. $\lim_{x \rightarrow 0} (\cos x)^{ctgx^2}$

Variant 12

1. $\lim_{x \rightarrow 1} \frac{2^{\ln x} - x}{x-1}$
2. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{tg 3x}{tg 5x}$
3. $\lim_{x \rightarrow 1} (x-1)^{x-1}$

Variant 13

1. $\lim_{x \rightarrow \infty} \frac{5x^4 - 3x^3 + 2x}{2x^4 + 4x^2 + 7}$
2. $\lim_{x \rightarrow \pi} (\pi - x) \cdot tg \frac{x}{2}$
3. $\lim_{x \rightarrow 1} x^{\frac{1}{x-1}}$

Variant 14

1. $\lim_{x \rightarrow 0} \frac{\arcsin 3x}{1 - e^{3x}}$
2. $\lim_{x \rightarrow 0} \frac{tgx - x}{x - \sin x}$
3. $\lim_{x \rightarrow 0} (1 - \sin 2x)^{ctgx}$

Variant 15

1. $\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{tg 2x}$
2. $\lim_{x \rightarrow 1} \frac{2^{\ln x} - x}{1 - x}$
3. $\lim_{x \rightarrow 0} \left(\frac{x+2}{2-x} \right)^{\frac{1}{\sqrt{x}}}$

Variant 16

1. $\lim_{x \rightarrow 0} \frac{e^{-x} + \sin x - 1}{3x^2}$
2. $\lim_{x \rightarrow \pi} (\sin x)^{ctgx}$
3. $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{tgx}$

Variant 17

1. $\lim_{x \rightarrow 0} (\ln(x+e))^{\frac{1}{x}}$

2. $\lim_{x \rightarrow 0} \frac{\operatorname{tg} 3x - 2x}{\operatorname{tg} 5x + 4x}$

3. $\lim_{x \rightarrow \infty} \left(\cos \frac{4}{x} \right)^x$

Variant 18

1. $\lim_{x \rightarrow 0} \frac{1 - e^{2x}}{\operatorname{tg} x}$

2. $\lim_{x \rightarrow 0} \frac{1 - \cos 7x}{\operatorname{tg}^2 2x}$

3. $\lim_{x \rightarrow 0} (\cos x)^{\operatorname{ctg} x^2}$

Variant 19

1. $\lim_{x \rightarrow \infty} \frac{5x^5 - 3x^3 + 2x}{5x^4 + 3x^2 + 9x}$

2. $\lim_{x \rightarrow 1} \frac{\ln x}{1 - x^3}$

3. $\lim_{x \rightarrow 0} (\operatorname{ctg} 2x)^{\sin x}$

Variant 20

1. $\lim_{x \rightarrow 0} \frac{\arcsin 3x}{\operatorname{tg} 4x}$

2. $\lim_{x \rightarrow 0} x \ln x$

3. $\lim_{x \rightarrow 1} (1-x)^{\ln x}$

5. The Application of Differential Calculus

5.1. Increasing and Decreasing of a Function. Local Extremum of a Function

Let $x_1 < x_2$ and $f(x_1) < f(x_2)$ for any two points x_1, x_2 from the interval $(a; b)$. Then the function $f(x)$ increases in the interval $(a; b)$.

Let $x_1 < x_2$ and $f(x_1) > f(x_2)$ for any two points x_1, x_2 from the interval $(a; b)$. Then the function $f(x)$ decreases in the interval $(a; b)$.

Signs of increasing and decreasing of a function:

1) if $f'(x) > 0$ for all $x \in (a; b)$ then the function $f(x)$ increases on $(a; b)$;

2) if $f'(x) < 0$ for all $x \in (a; b)$ then the function $f(x)$ decreases on $(a; b)$.

A function $f(x)$ possesses a **local maximum** at a point x_0 if there exists a vicinity $0 < |x - x_0| < \delta$ of the point x_0 from the domain of a function and $f(x) < f(x_0)$ for all points from this vicinity.

A function $f(x)$ possesses a **local minimum** at a point x_0 if there exists a vicinity $0 < |x - x_0| < \delta$ of the point x_0 from the domain of a function and $f(x) > f(x_0)$ for all points from this vicinity.

Figures 3 and 4 show the geometric meaning of the previous definitions.

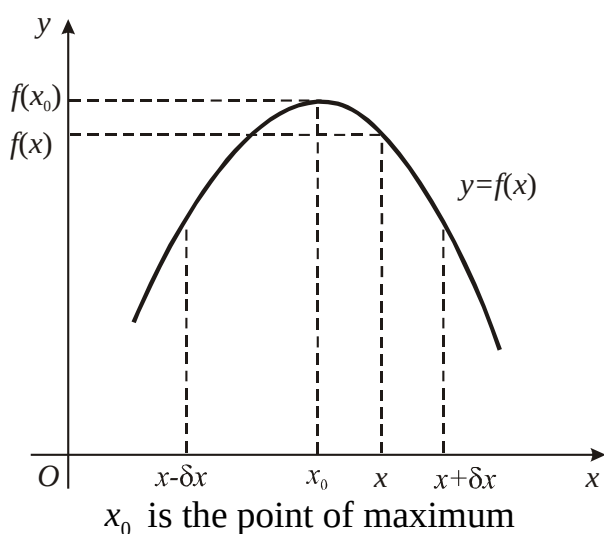


Fig. 3.

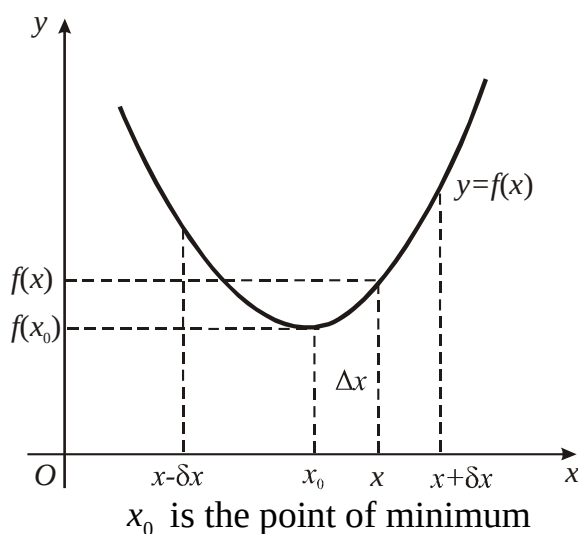


Fig. 4.

Let us find out the conditions for the existence of a local extremum.

Theorem 1 (A necessary condition for the existence of a local extremum) *If the function $f(x)$ has a local extremum and is differentiable at the point x_0 then $f'(x_0) = 0$.*

Consider the geometric meaning of Theorem 1. If the function $f(x)$ has a local extremum and is differentiable at the point x_0 then there is a tangent to the graph of the function $y = f(x)$ at this point and this tangent is parallel to the axis Ox . The condition of Theorem 1 is not a *sufficient* condition. For example, the derivative of the function $y = x^3$ at the point $x = 0$ is equal to zero, but the function has no extremum at this point.

Points at which the first derivative is zero are called critical points.

Theorem 2 (The first sufficient condition for the existence of a local extremum) *Let x_0 is a critical point of the function $f(x)$ and the function $f(x)$ is continuous at this point. Let the function $f(x)$ has a derivative $f'(x)$ in this vicinity except possibly the point x_0 . Then:*

- 1) if $f'(x) > 0$ in the interval $(x_0 - \delta; x_0)$ and $f'(x) < 0$ in the interval $(x_0; x_0 + \delta)$ then x_0 is the point of the local maximum of the function $f(x)$;
- 2) if $f'(x) < 0$ in the interval $(x_0 - \delta; x_0)$ and $f'(x) > 0$ in the interval $(x_0; x_0 + \delta)$ then x_0 is the point of the local minimum of the function $f(x)$;
- 3) if $f'(x)$ has the same sign on both intervals $(x_0 - \delta; x_0)$ and $(x_0; x_0 + \delta)$, then the function $f(x)$ has no extremum at the point x_0 .

Theorem 3 (The second sufficient condition for the existence of a local extremum) *Let x_0 is a critical point of the function $f(x)$, i.e. $f'(x_0) = 0$. Let in the vicinity of the point x_0 there exists a second continuous derivative and $f''(x_0) \neq 0$. Then the point x_0 is the point of local minimum if $f''(x_0) > 0$ and the point x_0 is the point of local maximum if $f''(x_0) < 0$.*

5.2. Finding an Extremum of a Function

To determine the local extremum of the function $f(x)$ we need:

- 1) find critical points of the function $f(x)$, i.e. solve the equation $f'(x_0) = 0$. Select those points that are internal points of the domain of a function. Then find points at where the derivative does not exist;
- 2) investigate the sign of the derivative in each of the intervals between critical points;
- 3) determine the maximum and minimum points of the function by determining the sign of the derivative $f'(x)$ change when passing through critical points from left to right. Calculate function values at these points.

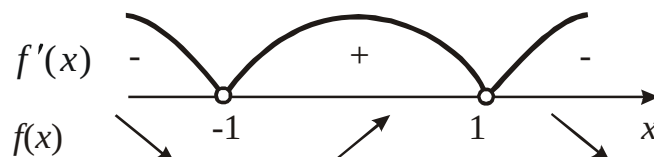
Example 1 Find intervals of monotony and the local extrema of the function

$$f(x) = x - \frac{x^5}{5}.$$

Solution $(-\infty; +\infty)$ is the domain of the function. Find the critical points:

$$f'(x) = 1 - \frac{5}{5}x^4 = 1 - x^4 = (1 - x^2)(1 + x^2) = (1 - x)(1 + x)(1 + x^2) = 0.$$

So, points $x = \pm 1$ are the critical points. Now let us find the signs of the derivative on each of the intervals:



Hence, if $x \in (-\infty; -1) \cup (1; \infty)$ then the function is decreasing. If $x \in (-1; 1)$ then the function is increasing. According to Theorem 2 $x = -1$ is the point of the local minimum and $x = 1$ is the point of the local maximum. Then

$$y_{\min} = y(-1) = -1 - \frac{(-1)^5}{5} = -\frac{4}{5}; \quad y_{\max} = y(1) = 1 - \frac{(1)^5}{5} = \frac{4}{5}.$$

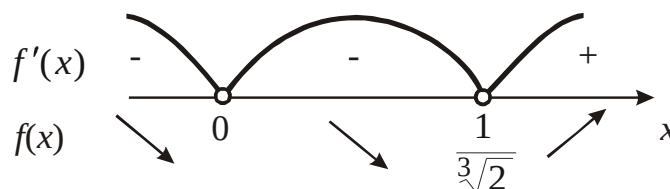
Example 2 Find intervals of monotony and the local extrema of the function

$$f(x) = x^2 + \frac{1}{x}.$$

Solution $(-\infty; 0) \cup (0; \infty)$ is the interval on which the function is defined (domain). The derivative

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = \frac{2(x^3 - \frac{1}{2})}{x^2} = \frac{2(x - \frac{1}{\sqrt[3]{2}})(x^2 + \frac{x}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{4}})}{x^2} = 0.$$

$f'(x)$ is equal to zero at the point $x = \frac{1}{\sqrt[3]{2}}$ and does not exist at the point $x = 0$. Find the signs of the derivative on each of the intervals:



If $x \in (-\infty; 0) \cup (0; \frac{1}{\sqrt[3]{2}})$ then the function is decreasing. If $x \in (\frac{1}{\sqrt[3]{2}}; \infty)$ then the function is increasing. Moreover, $x = \frac{1}{\sqrt[3]{2}}$ is the point of the local minimum. Then

$$y_{\min} = y\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{1}{\sqrt[3]{4}} + \frac{1}{\frac{1}{\sqrt[3]{2}}} = \frac{1}{\sqrt[3]{4}} + \sqrt[3]{2} = \frac{1 + \sqrt[3]{8}}{\sqrt[3]{4}} = \frac{1 + 2}{\sqrt[3]{4}} = \frac{3}{\sqrt[3]{4}} = \frac{3\sqrt[3]{2}}{4}.$$

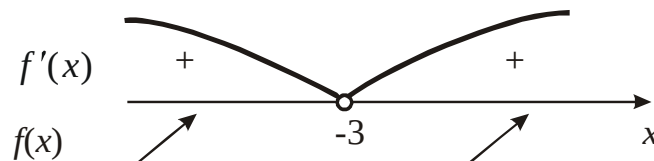
Note that the point $x = 0$ is not a critical point because at this point the function is not defined.

Example 3 Find function monotonicity interval and the local extrema of the function $f(x) = \frac{x-1}{x+3}$.

Solution $(-\infty; -3) \cup (-3; \infty)$ is the domain of the function is defined. Find the derivative

$$f'(x) = \frac{(x-1)'(x+3) - (x-1)(x+3)'}{(x+3)^2} = \frac{1(x+3) - 1(x-1)}{(x+3)^2} = \frac{x+3-x+1}{(x+3)^2} = \frac{4}{(x+3)^2}.$$

$f'(x) \neq 0$ for all x . Draw only a point $x = -3$ on the axis. The function is increasing on the whole numerical axis.



The function is not defined at the point $x = -3$.

Example 4 Find the local extrema of the function $y = 2x^3 + 3x^2 - 1$.

Solution $(-\infty; \infty)$ is the interval on which the function is defined. Derivative $y' = 6x^2 + 6x$. Solve the equation $y' = 0$. $6x^2 + 6x = 0 \Rightarrow 6x(x+1) = 0 \Rightarrow x = -1, x = 0$ are the critical points. Find the second derivative $y'' = 12x + 6$.

Find the sign of the derivative y'' at the critical points: $y''(-1) = 12(-1) + 6 = -6 < 0$; $y''(0) = 12 \cdot 0 + 6 = 6 > 0$. By the Theorem 3 $x = -1$ is the point of maximum and $x = 0$ is the point of minimum, moreover $y(-1) = 2(-1)^3 + 3(-1)^2 - 1 = -2 + 3 - 1 = 0$; $y(0) = 2(0)^3 + 3(0)^2 - 1 = -1$.

Individual Task 10

Find intervals of monotony and the local extrema of the function.

Variant 1
 $y = 0,5x^4 - 4x^2$

Variant 2
 $y = \frac{x}{x^2 + 1}$

Variant 3
 $y = \frac{3x}{x^2 + 4x + 4}$

Variant 4
 $y = \frac{1}{(x-1)(x-4)}$

Variant 5
 $y = e^{-x} - e^{-2x}$

Variant 6
 $y = \frac{5x}{4 - x^2}$

Variant 7
 $y = \frac{4 - 2x}{1 - x^2}$

Variant 8
 $y = \frac{x + 1}{(x - 1)^2}$

Variant 9
 $y = x^2 + \frac{1}{x^2}$

Variant 10
 $y = 3x^2 - \frac{1}{x}$

Variant 11
 $y = \frac{x^2 - 4}{2x^3}$

Variant 12
 $y = \frac{x^2}{(x - 2)}$

Variant 13
 $y = \frac{x^2}{x^3 - 4}$

Variant 14
 $y = x - \frac{2}{x}$

Variant 15
 $y = \frac{2x}{1 + x^2}$

Variant 16
 $y = x + \sqrt{3 - x}$

Variant 17
 $y = \frac{x^3 + 1}{x^2}$

Variant 18
 $y = \frac{x}{x - 1}$

Variant 19
 $y = \frac{x^2}{x - 1}$

Variant 20
 $y = \frac{x^3}{x^2 - 4}$

5.3. The Largest and the Smallest Value of a Function on an interval (Global Extrema)

To determine the largest and smallest value of the function $y = f(x)$ on the segment $[a; b]$ we need:

- 1) find $f'(x)$ and critical points of the function $f(x)$;
- 2) calculate the value of the function at critical points which belong to the interval $[a; b]$ and at the points a, b ;
- 3) choose the largest and smallest value among the obtained ones.

Example 1. Find the largest and smallest value of the function $f(x) = x^3 - 3x^2 + 3x + 2$ on the segment $[-2; 2]$.

Solution. Find the critical points from the condition $f'(x) = 0$:
 $f'(x) = 3x^2 - 6x + 3 = 0; \Rightarrow 3x^2 - 6x + 3 = 0; \Rightarrow (x - 1)^2 = 0 \Rightarrow x_{1,2} = 1$. This point belongs to the segment $[-2; 2]$. We have

$$f(-2) = (-2)^3 - 3(-2)^2 + 3(-2) + 2 = -8 - 12 - 6 + 2 = -24;$$

$$f(1) = 1^3 - 3 \cdot 1^2 + 3 \cdot 1 + 2 = 1 - 3 + 3 + 2 = 3;$$

$$f(2) = 2^3 - 3 \cdot 2^2 + 3 \cdot 2 + 2 = 8 - 12 + 6 + 2 = 4.$$

$$\text{Hence, } f_{\inf} = f(-2) = -24 \quad f_{\sup} = f(2) = 4.$$

Example 2. Find the largest and smallest value of the function $f(x) = x^2 \ln x$ on the segment $[1; e]$.

Solution. Find the derivative

$$f'(x) = (x^2 \ln x)' = (x^2)' \ln x + x^2 (\ln x)' = 2x \ln x + \frac{x^2}{x} =$$

$= 2x \ln x + x = x(2 \ln x + 1)$. Find the critical points from the condition $f'(x) = 0$, i.e. $x(2 \ln x + 1) = 0$. $x > 0$ for the function $\ln x$, then $x \neq 0$. So, find the critical points

from the condition $2\ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$. This point does not belong to the segment $[1; e]$. Calculate the values of the function at the ends of the segment, i.e. at the points $x = 1$ and $x = e$.

$$f(1) = 1^2 \ln 1 = 0; \quad f(e) = e^2 \ln e = e^2. \text{ Hence, } f_{\sup} = f(e) = e^2, \quad f_{\inf} = f(1) = 0.$$

Example 3. Find the largest and smallest value of the function $f(x) = x + \cos^2 x$ on the segment $\left[0; \frac{\pi}{2}\right]$.

Solution. Find the critical points:

$$f'(x) = (x + \cos^2 x)' = 1 + 2\cos x(-\sin x) = 1 - \sin 2x = 0;$$

$$\sin 2x = 1; \quad 2x = \frac{\pi}{2} + 2\pi n, \quad n \in \mathbb{Z}; \quad x = \frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}.$$

Choose the points which belong to the segment $\left[0; \frac{\pi}{2}\right]$. It is the point $x = \frac{\pi}{4}$.

Find the value of the function at the ends of the segment and at this point.

$$f(0) = 0 + \cos^2 0 = 1; \quad f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \cos^2\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4} + \frac{2}{4} = \frac{\pi + 2}{4};$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cos^2\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \Rightarrow f_{\inf} = f(0) = 1, \quad f_{\sup} = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Individual Task 11

Find the largest and smallest value of the function in the specified segment.

Variant 1

$$y = 6x^5 - 15x^4 + 10x^3 + 2, \\ x \in [0; 3]$$

Variant 2

$$y = 2\sin x + \cos 2x, \\ x \in \left[0; \frac{\pi}{2}\right]$$

Variant 3

$$y = \operatorname{tg} x + \operatorname{ctg} 2x, \\ x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$$

Variant 4

$$y = 2x^2 - \ln x, \\ x \in [1; e]$$

Variant 5

$$y = x + \frac{8}{x^4}, \\ x \in [1; 3]$$

Variant 6

$$y = \ln(x^2 - 3x + 4) \\ x \in [0; 3]$$

Variant 7

$$y = (x - 1) \cdot e^{-x}, \\ x \in [0; 3]$$

Variant 8

$$y = \frac{x^5 - 8}{x^4}, \\ x \in [-3; 1]$$

Variant 9

$$y = 3x^4 - 16x^3 + 2, \\ x \in [-3; 1]$$

Variant 10

$$y = \sqrt{x - x^3}, \\ x \in [-2; 2]$$

Variant 11

$$y = (x - 1) \cdot e^{-x}, \\ x \in [0; 3]$$

Variant 12

$$y = (x + 1)\sqrt[3]{x^2}, \\ x \in [-1; 3]$$

Variant 13

$$y = \frac{1 + \ln x}{x}, \\ x \in \left[\frac{1}{e}; e\right]$$

Variant 14

$$y = \frac{\ln x}{x}, \\ x \in [1; e]$$

Variant 15

$$y = \frac{3x}{x^2 + 1}, \\ x \in [0; 3]$$

Variant 16

$$y = x \ln x \\ x \in \left[\frac{1}{e^2}; 1\right]$$

Variant 17
 $y = x^3 \cdot \leq e^{x+1},$
 $x \in [-4; 0]$

Variant 18
 $y = \ln(x^2 - 3x + 4)$
 $x \in [0; 3]$

Variant 19
 $y = \frac{x^2 - 2x + 2}{x - 1},$
 $x \in [1.5; 3]$

Variant 20
 $y = e^{6x - x^2},$
 $x \in [-3; 3]$

5.4. Convexity and Concavity of Curves. Inflection Points

The curve $y = f(x)$ is convex in the interval $(a; b)$, if all its points, except the point of contact, lie below the tangent to the curve in this interval (fig. 5). The curve $y = f(x)$ is concave in the interval $(a; b)$, if all its points, except the point of contact, lie higher the tangent to the curve in this interval (fig. 6). An inflection point is a point on a curve that separates its convex part from the concave (fig. 7).

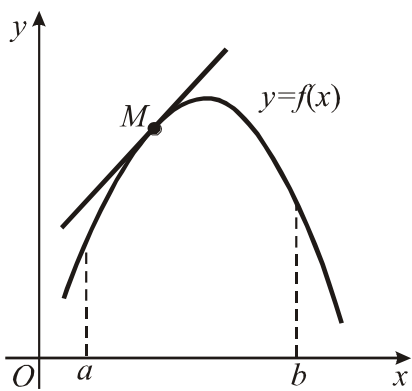


Fig. 5

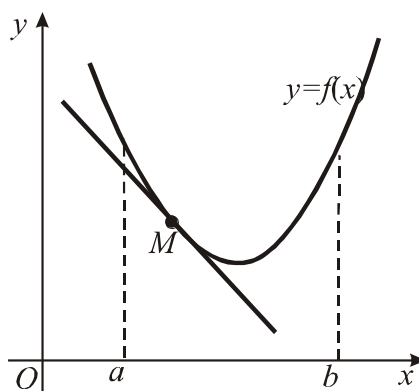


Fig. 6

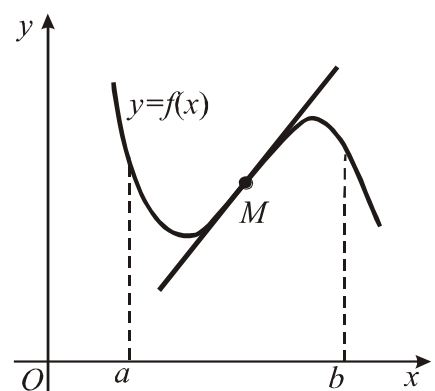


Fig. 7

Theorem 4 Let the function $y = f(x)$ be a twice differentiable function in the interval $(a; b)$. Then:

- 1) if $f''(x) < 0, x \in (a; b)$, then the function graph is convex in the interval $(a; b)$;
- 2) if $f''(x) > 0, x \in (a; b)$, then the function graph is concave in the interval $(a; b)$.

A necessary condition for the existence of an inflection point follows from Theorem 4. The points at which the second derivative $f''(x)$ is equal to zero or does not exist are called the second kind of critical points of a function $y = f(x)$.

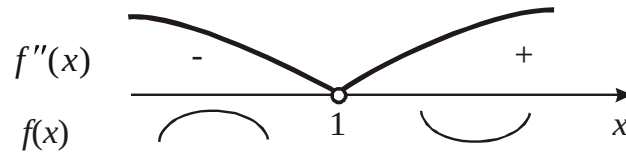
We formulate sufficient conditions for the existence of inflection points.

Theorem 5 Let x_0 be a second-kind critical point of the function $y = f(x)$. If the second derivative $f''(x)$ changes sign when passing through the point x_0 then the point $(x_0 ; f(x_0))$ is the inflection point of the curve $y = f(x)$.

Example 1. Find the convexity and concavity intervals and the inflection points of the curve $y = x^3 - 3x^2 + 9x + 6$.

Solution. Find $y' = 3x^2 - 6x + 9; y'' = 6x - 6$.

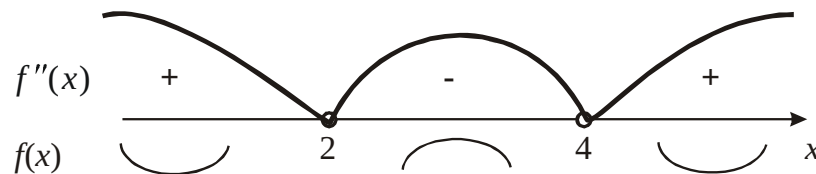
Solve the equation $y''(x)=0$; $6x-6=0$; $6(x-1)=0$; $x=1$,
i.e. $x=1$ is a second-kind critical point. Find the sign of the second derivative at each of the intervals.



If $x < 1$ then $f''(x) < 0$ and the curve is convex. If $x > 1$ then $f''(x) > 0$ and the curve is concave. When passing through the point $x=1$, the second derivative changes sign. Hence, the point $(1;13)$ is the inflection point of the curve.

Example 2. Find the convexity and concavity intervals and the inflection points of the curve $y = x^4 - 12x^3 + 48x^2 - 50$.

Solution. Find the derivatives: $y' = 4x^3 - 36x^2 + 96x$; $y'' = 12x^2 - 72x + 96$. Find the second-kind critical points ($y''(x)=0$): $12x^2 - 72x + 96 = 0$; $x^2 - 6x + 8 = 0$; $(x-2)(x-4)=0$; $x_1 = 2$, $x_2 = 4$. Obtain a sign of the second derivative at the obtained intervals.



For $x \in (-\infty; 2) \cup (4; +\infty)$ $f''(x) > 0$ and the curve is concave. For $x \in (2; 4)$ $f''(x) < 0$ and the curve is convex. Find $y(2)$ and $y(4)$: $y(2) = 62$ and $y(4) = 206$. The points $(2;62)$ and $(4;206)$ are the inflection points of a given curve.

Example 3. Find the convexity and concavity intervals and the inflection points of the curve $y = x^4(12 \ln x - 7)$.

Solution. Find the derivatives: $y' = (x^4)'(12 \ln x - 7) + x^4(12 \ln x - 7)' = 4x^3(12 \ln x - 7) + x^4 \frac{12}{x} = 4x^3(12 \ln x - 7) + 12x^3 = 16x^3(3 \ln x - 1)$;
 $y'' = (16x^3(3 \ln x - 1))' = 48x^2(3 \ln x - 1) + 16x^3 \cdot \frac{3}{x} = 48x^2 \cdot 3 \ln x = 144x^2 \ln x$. We calculate $144x^2 \ln x = 0 \Rightarrow \begin{cases} x_1 = 0 \\ \ln x = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0, \\ x_2 = 1. \end{cases}$ We obtain a critical point $x=1$. Consider the function graph $y = \ln x$ (Fig. 8).

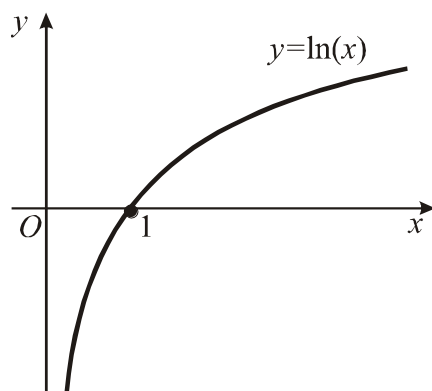
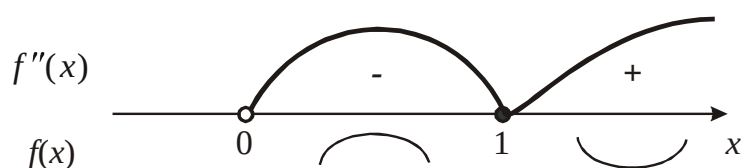


Fig. 8

We define the sign of the second derivative in the vicinity of the point $x = 1$.



If $x \in (0;1)$ then the curve is convex, if $x \in (1;\infty)$ then the curve is concave; $y(1) = -7$. The point $(1;7)$ is the inflection point of a given curve.

5.5. Asymptotes of a Curve

The straight line to which the curve point approaches infinitely when it is unboundedly distant from the origin is called the asymptote of this curve. We consider three types of asymptotes: vertical, horizontal, and inclined (Fig. 9-11).

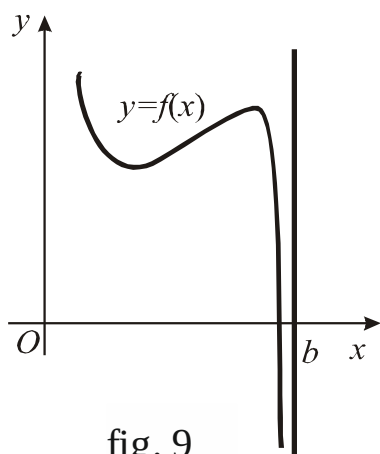


fig. 9

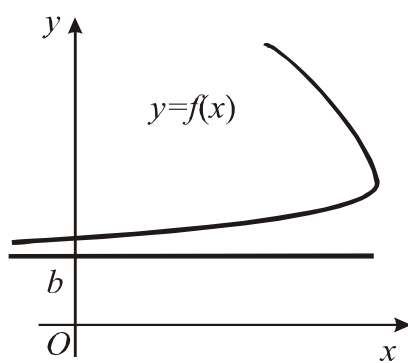


fig. 10

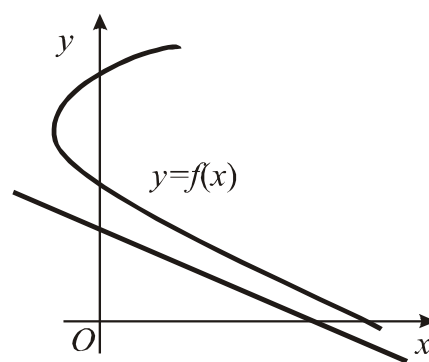


fig. 11

The line $x = c$ is the vertical asymptote (Fig. 9) if $\lim_{x \rightarrow c+0} f(x) = \pm\infty$ or $\lim_{x \rightarrow c-0} f(x) = \pm\infty$.

The line $y = kx + b$ is the inclined asymptote (Fig. 11) if finite limits

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = k \ (k \neq 0) \text{ and } \lim_{x \rightarrow \pm\infty} (f(x) - kx) = b \text{ exist.}$$

Note! We should consider both cases $x \rightarrow +\infty$ and $x \rightarrow -\infty$.

The line $y = b$ is the horizontal asymptote (Fig. 10) if finite limits

$$\lim_{x \rightarrow \pm\infty} f(x) = b \text{ exist.}$$

Example 1. Find the asymptotes of the curve $y = \frac{6x^4 + 3x^3}{5x^3 + 1}$.

Solution. We equate the denominator of the fraction to zero and find the discontinuity point of the function: $5x^3 + 1 = 0$; $x^3 = -\frac{1}{5}$; $x = -\sqrt[3]{\frac{1}{5}}$. The function has a discontinuity of the second kind at the point $x = -\sqrt[3]{\frac{1}{5}}$. Then the straight line

$x = -\sqrt[3]{\frac{1}{5}}$ is a vertical asymptote of the curve.

Then we find the equation of the inclined asymptote:

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{6x^4 + 3x^3}{x(5x^3 + 1)} = \lim_{x \rightarrow \infty} \frac{6x^4 + 3x^3}{5x^4 + x} = \lim_{x \rightarrow \infty} \frac{\frac{6x^4}{x^4} + \frac{3x^3}{x^4}}{\frac{5x^4}{x^4} + \frac{x}{x^4}} = \lim_{x \rightarrow \infty} \frac{6 + \frac{3}{x}}{5 + \frac{1}{x^3}} = \frac{6}{5};$$

$$b = \lim_{x \rightarrow \infty} (f(x) - kx) = \lim_{x \rightarrow \infty} \left(\frac{6x^4 + 3x^3}{5x^3 + 1} - \frac{6}{5}x \right) = \lim_{x \rightarrow \infty} \frac{30x^4 + 15x^3 - 30x^4 - 6x}{5(5x^3 + 1)} =$$

$$= \lim_{x \rightarrow \infty} \frac{15x^3 - 6x}{5(5x^3 + 1)} = \lim_{x \rightarrow \infty} \frac{\frac{15x^3}{x^3} - \frac{6x}{x^3}}{\frac{25x^3}{x^3} + \frac{5}{x^3}} = \lim_{x \rightarrow \infty} \frac{15 - \frac{6}{x^2}}{25 + \frac{5}{x^3}} = \frac{3}{5};$$

$y = kx + b$; $y = \frac{6}{5}x + \frac{3}{5}$; $5y = 6x + 3$; $5y - 6x - 3 = 0$ is the equation of the inclined asymptote.

To find the horizontal asymptote we obtain $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{6x^4 + 3x^3}{5x^3 + 1} =$

$$= \lim_{x \rightarrow \infty} \frac{\frac{6x^4}{x^4} + \frac{3x^3}{x^4}}{\frac{5x^3}{x^4} + \frac{1}{x^4}} = \lim_{x \rightarrow \infty} \frac{6 + \frac{3}{x}}{\frac{5}{x} + \frac{1}{x^4}} = \infty, \text{ i.e. there are no horizontal asymptotes.}$$

5.6. Algorithm of a function investigation. Curve Sketching

To investigate the behavior of a function the following steps could be used:

- 1) find the domain of a function;
- 2) find the points of discontinuity and establish their characteristics;
- 3) find the intersection points of the function graph with the coordinate axes;
- 4) check if the function is periodic, even, or odd;
- 5) find the points of the local extremum, the value of the function at these points, as well as the intervals of monotonicity of the function;
- 6) find the inflection points, as well as the intervals of convexity and concavity;
- 7) find the asymptotes of the curve and investigate function behavior at infinitely distant points;
- 8) sketch the graph of a function.

Example 1. Investigate and sketch the function $y = \frac{3x^4 + 1}{x^3}$.

Solution. Follow the scheme mentioned above.

1. The interval on which the function is defined: $x \in (-\infty; 0) \cup (0; \infty)$.

2. The point of discontinuity is $x = 0$. Calculate

$\lim_{x \rightarrow 0-0} \frac{3x^4 + 1}{x^3} = \left\{ \frac{1}{-0} \right\} = -\infty$; $\lim_{x \rightarrow 0+0} \frac{3x^4 + 1}{x^3} = \left\{ \frac{1}{+0} \right\} = +\infty$, then $x = 0$ is the point of discontinuity of the second kind.

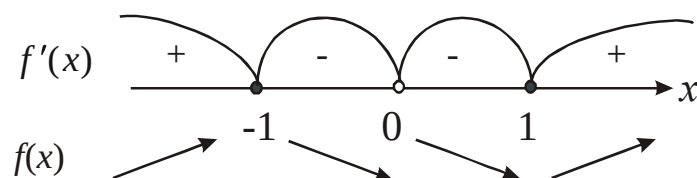
3. The graph does not cross the ordinate axis. To find intersection points of the graph with the abscissa axis, we should solve the equation: $y = 0$, $\frac{3x^4 + 1}{x^3} = 0$, $3x^4 + 1 = 0$. This equation has no real roots. The function does not cross the axis Ox .

4. The function is not periodic.

Check if the function is even or odd: $f(-x) = \frac{3(-x)^4 + 1}{(-x)^3} = \frac{3x^4 + 1}{-x^3}$,

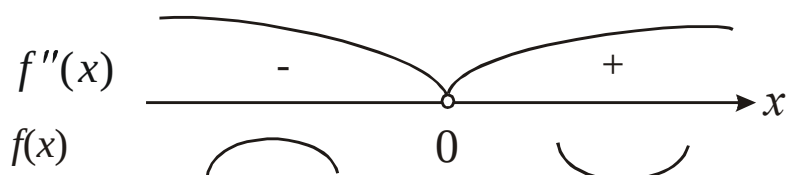
$f(-x) = -f(x)$ – the function is odd, its graph is symmetrical with respect to the origin.

5. Find the derivative: $y' = \left(\frac{3x^4 + 1}{x^3} \right)' = \frac{12x^3 \cdot x^3 - (3x^4 + 1)3x^2}{(x^3)^2} =$
 $= \frac{12x^6 - 9x^6 - 3x^2}{x^6} = \frac{3x^6 - 3x^2}{x^6} = \frac{3x^2(x^4 - 1)}{x^6} = \frac{3(x^2 - 1)(x^2 + 1)}{x^4} =$
 $= \frac{3(x - 1)(x + 1)(x^2 + 1)}{x^4}$. Solve the equation $y' = 0$, $3(x - 1)(x + 1)(x^2 + 1) = 0$
 $\Rightarrow x_1 = 1$; $x_2 = -1$. $x_1 = 1$, $x_2 = -1$, $x_3 = 0$ are the critical points. Study the sign of the derivative at each of the obtained intervals.



The function decreases in the interval $(-1, 0) \cup (0, 1)$ and the function increases in the interval $(-\infty; -1) \cup (1; \infty)$. There is a local maximum at the point $x_2 = -1$ (p. A, Fig. 12) and $y_{\max} = y(-1) = \frac{3(-1)^4 + 1}{(-1)^3} = \frac{3+1}{-1} = -4$. There is a local minimum at the point $x_1 = 1$ (p. B, Fig. 12) and $y_{\min} = y(1) = \frac{3(1)^4 + 1}{(1)^3} = \frac{3+1}{1} = 4$. The function is undefined at the point $x_3 = 0$.

6. Find the second derivative: $y'' = \left(\frac{3(x^4 - 1)}{x^4} \right)' = \frac{3(4x^3x^4 - (x^4 - 1)4x^3)}{(x^4)^2} =$
 $= \frac{3(4x^7 - 4x^7 + 4x^3)}{x^8} = \frac{12x^3}{x^8} = \frac{12}{x^5}.$



The curve is convex in the interval $(-\infty; 0)$ and the curve is concave in the interval $(0; \infty)$.

7. From 2 follows that the line $x = 0$ is a vertical asymptote.

Find the inclined asymptotes if they exist:

$$y = kx + b; \quad k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{\frac{3x^4 + 1}{x^3}}{x} = \lim_{x \rightarrow \pm\infty} \frac{3x^4 + 1}{x^4} = 3;$$

$$b = \lim_{x \rightarrow \pm\infty} (f(x) - kx) = \lim_{x \rightarrow \pm\infty} \left(\frac{3x^4 + 1}{x^3} - 3x \right) = \lim_{x \rightarrow \pm\infty} \frac{3x^4 + 1 - 3x^4}{x^3} = \lim_{x \rightarrow \pm\infty} \frac{1}{x^3} = \pm 0, \text{ i.e. } y = 3x$$

is the inclined asymptote. There are no other asymptotes.

8. Plot the function (Fig. 12).

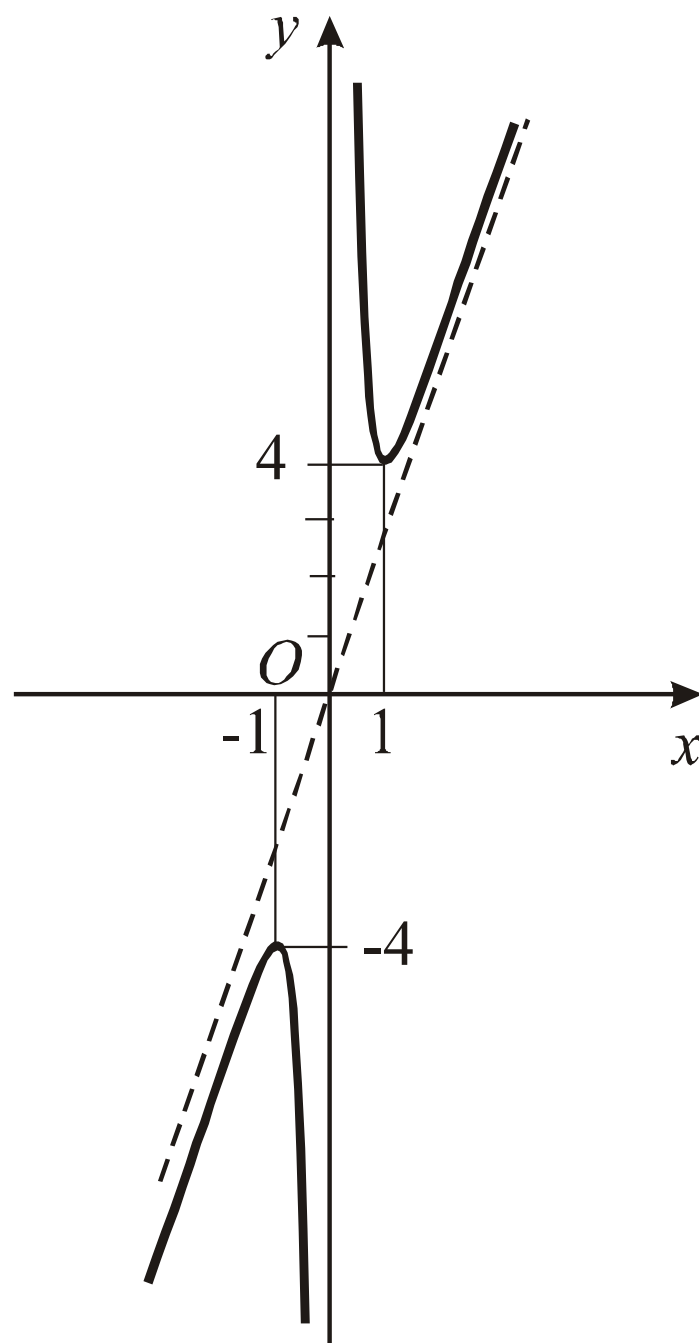


Fig. 12

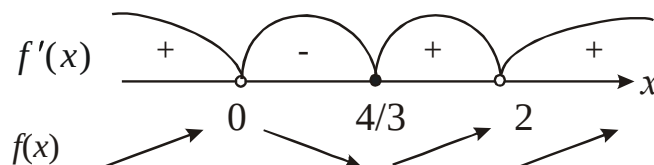
Example 2. Investigate and sketch the function $y = \sqrt[3]{x^3 - 2x^2}$.

Solution. Follow the above scheme.

1. The interval in which the function is defined is all the abscissa axis.
2. There are no points of discontinuity.
3. If $x = 0$, then $y = 0$. If $y = 0$ then $x = 0$ or $x = 2$. So the curve goes through points $B(0;0)$ and $A(2;0)$.
4. The function is not even or odd (indifferent). Thus, there is no symmetry.
5. Find the derivative:

$$y' = \left(\sqrt[3]{x^3 - 2x^2} \right)' = \frac{1}{3} (x^3 - 2x^2)^{-\frac{2}{3}} \cdot (3x^2 - 4x) = \frac{3x^2 - 4x}{3\sqrt[3]{(x^3 - 2x^2)^2}} = \frac{x(3x - 4)}{3\sqrt[3]{(x^2(x - 2))^2}} = \frac{x(x - 4/3)}{\sqrt[3]{x^4(x - 2)^2}}.$$

There are critical points $x = 0$; $x = 2$; $x = \frac{4}{3}$.



The function increases in the interval $(-\infty; 0) \cup (\frac{4}{3}; \infty)$ and the function decreases in the interval $(0; \frac{4}{3})$. The point $x = 0$ is a point of maximum and $y_{\max} = y(0) = 0$. Since the first derivative does not exist at the point $x = 0$, this is the point of maximum. The point $x = \frac{4}{3}$ is a point of minimum.

$$y_{\min} = y\left(\frac{4}{3}\right) = \sqrt[3]{\left(\frac{4}{3}\right)^2 \left(\frac{4}{3} - 2\right)} = \sqrt[3]{-\frac{32}{27}} = -\frac{2\sqrt[3]{4}}{3} \approx -1,1.$$

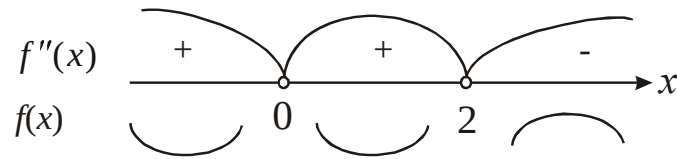
The point $x = 2$ is not an extremum point.

6. Find the second derivative:

$$\begin{aligned} y'' &= \left(\frac{x(x - \frac{4}{3})}{\sqrt[3]{x^4(x - 2)^2}} \right)' = \left(\frac{x(x - \frac{4}{3})}{x \cdot \sqrt[3]{x(x - 2)^2}} \right)' = \left(\frac{x - \frac{4}{3}}{\sqrt[3]{x(x - 2)^2}} \right)' = \\ &= \frac{\sqrt[3]{x(x - 2)^2} - (x - \frac{4}{3}) \frac{1}{3} (x(x - 2)^2)^{-\frac{2}{3}} ((x - 2)^2 + 2x(x - 2))}{\left(\sqrt[3]{x(x - 2)^2} \right)^2} = \\ &= \frac{\sqrt[3]{x(x - 2)^2} - (x - \frac{4}{3}) \frac{((x - 2)^2 + 2x(x - 2))}{3\sqrt[3]{x^2(x - 2)^4}}}{\sqrt[3]{x^2(x - 2)^4}} = \end{aligned}$$

$$\begin{aligned}
&= \frac{3\sqrt[3]{x^3(x-2)^6} - \frac{(3x-4)((x-2)(x-2+2x))}{3}}{3 \cdot \sqrt[3]{x^4(x-2)^8}} = \frac{9\sqrt[3]{x^3(x-2)^6} - (3x-4)(x-2)(3x-2)}{9 \cdot \sqrt[3]{x^4(x-2)^8}} = \\
&= \frac{(x-2)(9x(x-2) - (3x-4)(3x-2))}{9 \cdot \sqrt[3]{x^4(x-2)^8}} = \frac{(x-2)(9x^2 - 18x - 9x^2 + 6x + 12x - 8)}{9 \cdot \sqrt[3]{x^4(x-2)^8}} = \\
&= -\frac{8}{9} \frac{(x-2)}{\sqrt[3]{x^4(x-2)^8}}.
\end{aligned}$$

The second derivative does not exist when $x = 0$ and $x = 2$.



The curve is concave in the interval $(-\infty; 0) \cup (0; 2)$ and the curve is convex in the interval $(2; \infty)$.

7. There are no vertical asymptotes.

Find the inclined asymptotes: $y = kx + b$; $k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{\sqrt[3]{x^3 - 2x^2}}{x} = 1$;

$$\begin{aligned}
b &= \lim_{x \rightarrow \pm\infty} (f(x) - kx) = \lim_{x \rightarrow \pm\infty} (\sqrt[3]{x^3 - 2x^2} - x) = \{\infty - \infty\} = \\
&= \lim_{x \rightarrow \pm\infty} \frac{(\sqrt[3]{x^3 - 2x^2} - x)(\sqrt[3]{(x^3 - 2x^2)^2} + x\sqrt[3]{x^3 - 2x^2} + x^2)}{(\sqrt[3]{(x^3 - 2x^2)^2} + x\sqrt[3]{x^3 - 2x^2} + x^2)} = \\
&= \lim_{x \rightarrow \pm\infty} \frac{x^3 - 2x^2 - x^3}{(\sqrt[3]{(x^3 - 2x^2)^2} + x\sqrt[3]{x^3 - 2x^2} + x^2)} = -\frac{2}{3}.
\end{aligned}$$

Then $y = x - \frac{2}{3}$ is an inclined asymptote.

8. Plot the function (Fig. 13).

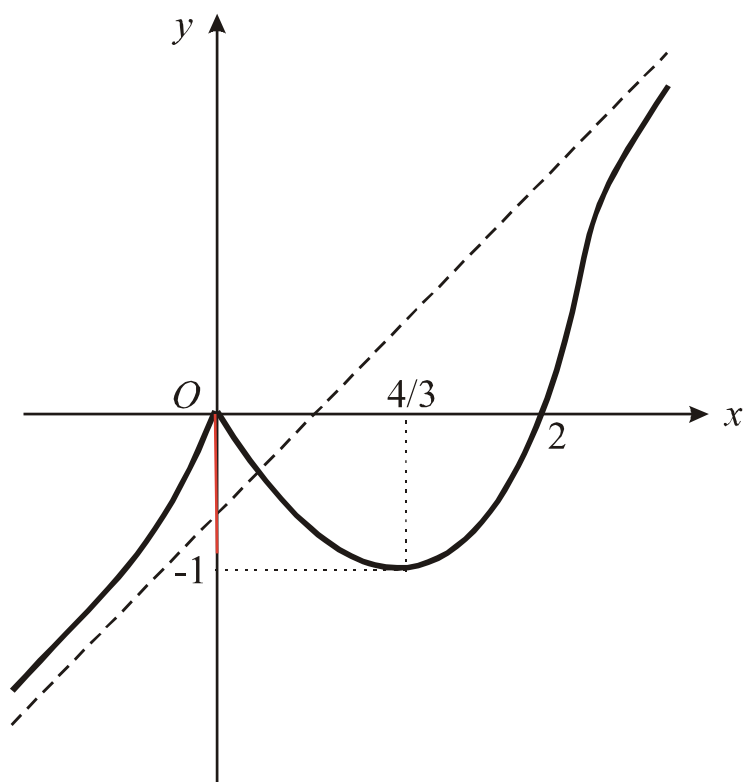


Fig. 13

Individual Task 12

Investigate and sketch the function.

Variant 1

a) $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + 2;$

b) $f(x) = \frac{x}{x^2 - 1}.$

Variant 3

a) $f(x) = \frac{1}{8}(12x - x^3);$

b) $f(x) = \frac{(x-1)^2}{x^2}.$

Variant 5

a) $f(x) = 2x^3 + 3x^2 - 1;$

b) $f(x) = \frac{x+1}{x^2}.$

Variant 2

a) $f(x) = x^4 - 6x^2 + 5;$

b) $f(x) = \frac{2(x-3)}{(x-2)^2}.$

Variant 4

a) $f(x) = x^3 - 3x + 2;$

b) $f(x) = \frac{4(x-1)}{(x+1)^2}.$

Variant 6

a) $f(x) = \frac{1}{9}(x^4 - 4x^3);$

b) $f(x) = \left(\frac{x+1}{x-1}\right)^2.$

Variant 7

a) $f(x) = 3 + 2x^2 - x^4$;

b) $f(x) = \frac{x^3}{(x-1)^2}$.

Variant 9

a) $f(x) = \frac{3}{16}x^5 - \frac{5}{4}x^3$;

b) $f(x) = \frac{2x^2 - 18}{x + 2}$.

Variant 11

a) $f(x) = 3x^4 - 4x^3 + 1$;

b) $f(x) = \frac{x^3}{(x+1)^2}$.

Variant 13

a) $f(x) = -\frac{1}{9}(x^4 + 4x^3)$;

b) $f(x) = \frac{4}{x^2 + 2x - 2}$.

Variant 15

a) $f(x) = 4x^2 - 2x^4$;

b) $f(x) = \frac{x^3 + 1}{x}$.

Variant 17

a) $f(x) = 3 + 2x^2 - x^4$;

b) $f(x) = \frac{x^2}{x^2 - 1}$.

Variant 19

a) $f(x) = 2x^3 + 3x^2 - 1$;

b) $f(x) = \frac{(x+1)^2}{x^2}$.

Variant 8

a) $f(x) = x^2 + \frac{1}{3}x^3$;

b) $f(x) = \frac{-8x}{x^2 + 1}$.

Variant 10

a) $f(x) = x^3 - 3x^2$;

b) $f(x) = \frac{x^2 + 2x - 3}{x}$.

Variant 12

a) $f(x) = x^4 - 4x^3$;

b) $f(x) = \frac{2x^2 + x + 1}{x + 1}$.

Variant 14

a) $f(x) = 2x^3 + 3x^2$;

b) $f(x) = \frac{2x^2 + 16}{x^2}$.

Variant 16

a) $f(x) = 3x^2 - 2x^3 - 1$;

b) $f(x) = \frac{(x-3)^2}{2x}$.

Variant 18

a) $f(x) = -\frac{1}{6}(x^2 - 4)^2$;

b) $f(x) = \frac{4(3-x)}{(x-2)^2}$.

Variant 20

a) $f(x) = \frac{1}{9}(x^4 - 4x^3)$;

b) $f(x) = \frac{x^2 - 3x + 3}{x}$.

6. Specific Tasks

In this section, the application of the above theory for solving specific problems is considered.

Example 1. There are boards available that can be used to build a 200 m long fence. It is necessary to protect the rectangle of the yard of the largest area using the wall of a nearby house for one side of the yard (Fig. 14).

Solution. Denote by x the length of those sides of the fence that are perpendicular to the wall of the house. Then the length of the side which is parallel to the house is $200 - 2x$, and the area of the entire yard is $S = F = x(200 - 2x) = 200x - 2x^2$.

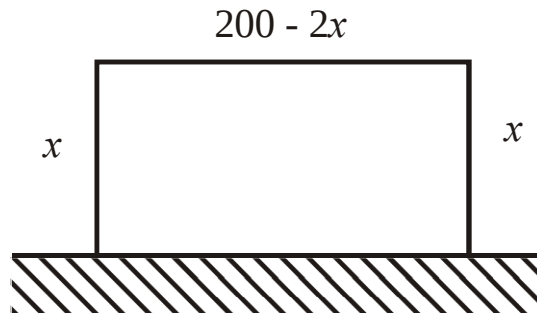


Fig. 14

Note that x changes in a segment $[0;100]$. The problem is reduced to determining the largest value of the function in the segment. Using 5.2, find $F' = 200 - 4x$. Putting $F' = 0$ find the critical point $200 - 4x = 0$ or $x = 50$. It is a point of maximum because $F'' = -4 < 0$. This point corresponds to F_{sup} . So, the size of the yard is 50×100 , and its area is equal to 5000 m^2 . If we take other sizes, for example, 45×110 or 55×90 , then we get a yard with a smaller area.

Example 2. From a round log of diameter d , it is necessary to cut a riser, which has a rectangular cross-section and can absorb the greatest load. What should be the size of the riser?

Solution. The riser is a structural member that works for compression. Therefore, it will absorb the greatest load when its cross-sectional area is largest. The task is to determine the rectangle of the largest area that can be entered into a circle of diameter d (Fig. 15).

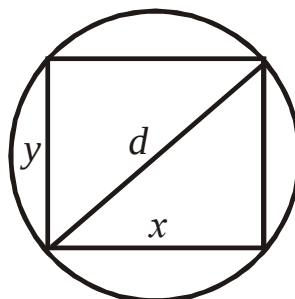


Fig. 15

Let x and y are the sides of the rectangle. Then $y = \sqrt{d^2 - x^2}$ and the area of the rectangle $S = S(x) = x\sqrt{d^2 - x^2}$, $0 < x < d$, a

$$S'(x) = \sqrt{d^2 - x^2} + x \frac{-2x}{2\sqrt{d^2 - x^2}} = \frac{d^2 - x^2 - x^2}{\sqrt{d^2 - x^2}} = \frac{d^2 - 2x^2}{\sqrt{d^2 - x^2}}.$$

$$S''(x) = \frac{-4x\sqrt{d^2 - x^2} - (d^2 - 2x^2) \frac{-2x}{\sqrt{d^2 - x^2}}}{d^2 - x^2} = \frac{-4x(d^2 - x^2) + x(d^2 - 2x^2)}{(d^2 - x^2)\sqrt{d^2 - x^2}} =$$

$$= \frac{x(2x^2 - 3d^2)}{(d^2 - x^2)^{3/2}}. \quad S'(x) = 0 \text{ when } x = \pm \frac{d}{\sqrt{2}} \text{ and does not exist when } x = \pm d. \text{ Since the}$$

function $S(x)$ is defined in the interval $(0; d)$, then it has a single critical point $x = \frac{d}{\sqrt{2}}$.

Find the second derivative and determine its sign at this point.

$$S''(x) = S''\left(\frac{d}{\sqrt{2}}\right) = \frac{\frac{d}{\sqrt{2}}(2\frac{d^2}{2} - 3d^2)}{(d^2 - \frac{d^2}{2})^{3/2}} = \frac{\frac{d}{\sqrt{2}}(-2d^2)}{\left(\frac{d^2}{2}\right)^{3/2}} = -\frac{2d^3}{\sqrt{2}\left(\frac{d^2}{2}\right)^{3/2}} < 0.$$

The function $S(x)$ reaches a maximum at this point. Then $y = \sqrt{d^2 - x^2} = \sqrt{d^2 - \frac{d^2}{2}} = \frac{d}{\sqrt{2}}$ and $S_{\max} = \frac{d^2}{2}$. So, the square riser with the section side equal to $\frac{d}{\sqrt{2}}$ assumes the greatest load.

Example 3. Find the dimensions of a can of volume V at which the least material will be used for its production.

Solution. Let the can have the shape of a cylinder with base radius r and height h . Then the full surface of the can is $S = 2\pi r^2 + 2\pi rh$. The volume of the can is $V = \pi r^2 h$, then $h = \frac{V}{\pi r^2}$ and $S = S(r) = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2(\pi r^2 + \frac{V}{r})$. Find the

smallest function value $S(r)$: $\frac{dS}{dr} = 2(2\pi r - \frac{V}{r^2}) = 2(\frac{2\pi r^3 - V}{r^2})$. If $2\pi r^3 - V = 0$ then

$$\frac{dS}{dr} = 0, \text{ i.e. the extremum point is } r = r_0 = \sqrt[3]{\frac{V}{2\pi}}.$$

Find the second derivative $\frac{d^2 S}{dr^2} = 2(2\pi + \frac{2V}{r^3})$.

$$r = r_0 \quad \left. \frac{d^2 S}{dr^2} \right|_{r=r_0} = 2(2\pi + \frac{2V}{r_0^3}) = 2(2\pi + \frac{2V 2\pi}{V}) = 12\pi > 0.$$

So, the function $S(r)$ has a minimum at $r = r_0$. Since $\lim_{r \rightarrow 0} S(r) = \infty$ and $\lim_{r \rightarrow \infty} S(r) = \infty$, this value is the least in the interval $(0; \infty)$. Find the height

$h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2} = 2r$. Thus, to have the smallest surface for a given volume of the cylinder, its height must be equal to its diameter.

Example 4. A vessel with a vertical wall of height h stands on a horizontal plane (Fig. 16). At what depth should the hole be placed so that the range of water discharge from the hole is the greatest (the fluid velocity that flows according to the Torricelli's law is $\sqrt{2gx}$, where x is hole placement depth, g is an acceleration of gravity)?

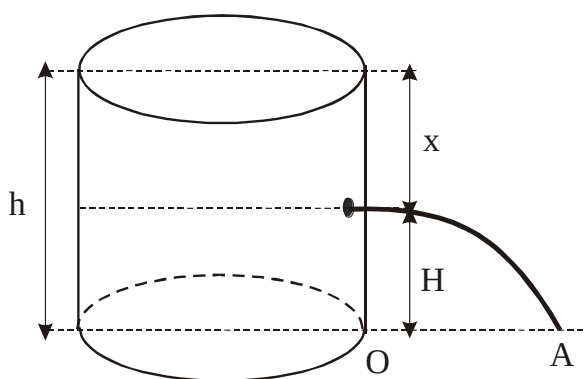


Fig. 16

Solution. We define H is the distance of the hole in the vessel from the horizontal plane and l is her water discharge ranging. Then $l = vt = \sqrt{2gx}t$ where t – a time of water discharge ranging from the hole to the plane. It is known that $H = \frac{gt^2}{2}$,

$$t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2(h-x)}{g}}, \text{ i.e. } l = l(x) = \sqrt{2gx} \sqrt{\frac{2(h-x)}{g}} = 2\sqrt{x(h-x)}, \quad 0 < x < h.$$

Find the largest value of the function $l(x)$: $l'(x) = \frac{h-2x}{\sqrt{x(h-x)}}$, $l'(x) = 0$ of $x = \frac{h}{2}$. As the

function $l(x)$ has the only critical point $x = \frac{h}{2}$, and by the condition of the problem there is such a position of the hole at which the maximum range of water out of the hole is the greatest, then this critical point is the desired.

Example 5. Let the light bulb move along a vertical line OB (Fig. 17). At what height from the horizontal plane do you need to place the light bulb so that at the given point A of the plane ($OA = a$) the illuminance is greatest?

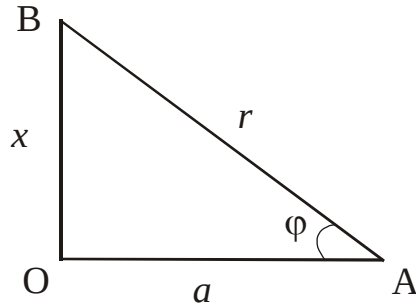


Fig. 17

Solution. It is known that the illuminance is $I = k \frac{\sin \phi}{r^2}$ where k is a proportionality coefficient, which depends on the light intensity of the bulb; $r = BA$ is the distance from the bulb to the point A . Let the desired height $OB = x$, then

$$\sin \phi = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}, \text{ and } I = I(x) = k \frac{\frac{x}{\sqrt{x^2 + a^2}}}{x^2 + a^2} = \frac{kx}{(x^2 + a^2)^{3/2}},$$

moreover, within the meaning of the problem $x \in (0; \infty)$.

We have

$$\begin{aligned} I'(x) &= \frac{k(x^2 + a^2)^{3/2} - kx \frac{3}{2}(x^2 + a^2)^{1/2} 2x}{(x^2 + a^2)^3} = \frac{k(x^2 + a^2)^{1/2}(x^2 + a^2 - 3x^2)}{(x^2 + a^2)^3} = \\ &= \frac{k(a^2 - 2x^2)}{(x^2 + a^2)^{5/2}}. \quad I'(x) = 0 \text{ of } x_{1,2} = \pm \frac{a}{\sqrt{2}}. \end{aligned}$$

The function $I(x)$ does not have other critical points. Since the interval $(0; \infty)$ contains only one critical point $x = \frac{a}{\sqrt{2}}$ and by the condition of the problem, there is a bulb position at which the illuminance at point A is greatest, then the desired height $OB = \frac{a}{\sqrt{2}}$.

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DERIVATIVE TABLE

1.	$C' = 0$	7a.	$(e^u)' = e^u \cdot u'$
2.	$x' = 1$	8.	$(\log_a u)' = \frac{1}{u} \log_a e \cdot u' \quad (a = \text{const})$
3.	$(u + v - w)' = u' + v' - w'$	8a.	$(\ln u)' = \frac{1}{u} \cdot u'$
4.	$(uv)' = u'v + uv'$	9.	$(\sin u)' = \cos u \cdot u'$
4a.	$(Cv)' = Cv'$	10.	$(\cos u)' = -\sin u \cdot u'$
5.	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	11.	$(\operatorname{tg} u)' = \frac{1}{\cos^2 u} \cdot u' = \sec^2 u \cdot u'$
5a.	$\left(\frac{C}{v}\right)' = -\frac{Cv'}{v^2}$	12.	$(\operatorname{ctg} u)' = -\frac{1}{\sin^2 u} \cdot u' = -\operatorname{cosec}^2 u \cdot u'$
5b.	$\left(\frac{u}{C}\right)' = \frac{u'}{C}$	13.	$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$
6.	$(u^\alpha)' = \alpha u^{\alpha-1} \cdot u' \quad (\alpha = \text{const})$	14.	$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$
6a.	$(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u'$	15.	$(\operatorname{arctg} u)' = \frac{1}{1+u^2} \cdot u'$
6b.	$\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$	16.	$(\operatorname{arcctg} u)' = -\frac{1}{1+u^2} \cdot u'$
7.	$(a^u)' = a^u \ln a \cdot u' \quad (a = \text{const})$	17.	$(u^v)' = vu^{v-1} \cdot u' + u^v \ln u \cdot v'$

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